FEM: Boundary value problems in several variables. An non-rigorous description.[1]

1.1 The heat equation

Let Ω be a solid in $\mathbf{R}^3$ with boundary surface Γ and let Ω_0 be an arbitrary subsolid of Ω with piecewise-smooth positively oriented boudary surface Γ_0. The **principle of energy conservation** states that the rate of change of the internal energy in Ω_0 equals to the net heat flux through Γ_0 plus the energy added through a heat source.

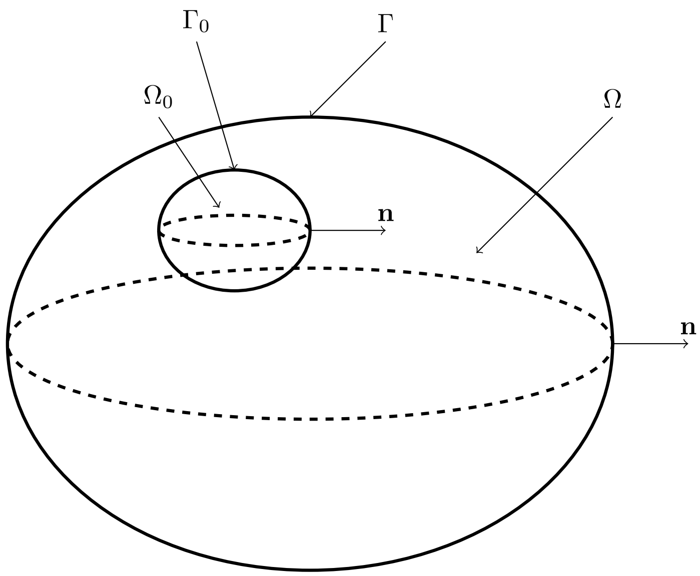

Figure 1: A solid Ω with a subsolid Ω_0.

Mathematically,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{\Omega_0} e\,\mathrm{d}V = -\iint\limits_{\Gamma_0} \mathbf{j}\cdot\mathbf{n}\,\mathrm{d}S + \iiint\limits_{\Omega_0} p\,\mathrm{d}V, \text{ for } t>0,$$

where $e = e(x,y,z,t)$ is the density of the internal energy ($[\mathtt{J/(m^3)}]$), $\mathbf{j} = \mathbf{j}(x,y,z,t)$ is the heat-flux density $[\mathtt{J/(m^3s)}]$ and p is the power density of the heat-source ($[\mathtt{J/(m^3s)}]$). We use the Divergence Theorem to transform the surface integral on the right to a triple integral, noting that, by definition :

$$\iint\limits_{\Gamma_0} \mathbf{j}\cdot\mathbf{n}\,\mathrm{d}S = \iint\limits_{\Gamma_0} \mathbf{j}\cdot\,\mathrm{d}\mathbf{S}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{\Omega_0} e\,\mathrm{d}V = -\iint\limits_{\Omega_0} \operatorname{div}\mathbf{j}\,\mathrm{d}V + \iiint\limits_{\Omega_0} p\,\mathrm{d}V, \text{ for } t>0.$$

[1]November 1, 2017, Mihály Kovács, Matematical Sciences, Chalmers tekniska högskola

Collecting all the terms on one side and interchanging the triple integral and the time derivative (this can be done, for example if e is continuously differentiable), we get

$$\iiint_{\Omega_0} (\partial_t e + \operatorname{div} \mathbf{j} - p)\, \mathrm{d}V = 0, \text{ for } t > 0.$$

If the integrand is continuous, this can only hold for every possible choice of Ω_0 if

$$\partial_t e + \operatorname{div} \mathbf{j} - p = 0, \quad \text{in } \Omega, \text{ for } t > 0,$$

or, using the ∇ notation,

$$\partial_t e + \nabla \cdot \mathbf{j} = p \quad \text{in } \Omega, \text{ for } t > 0. \tag{1}$$

In order to relate the internal energy e and the heat flux $\mathbf{j}$ to the temperature T (`[K]`) one needs further assumptions that are called **constitutive relations**. The first relation is that the internal energy is a linear function of the temperature:

$$e = e_0 + \sigma(T - T_0) = e_0 + \sigma u, \text{ with } u = T - T_0 \tag{2}$$

for some suitably chosen reference energy e_0 and temperature T_0. Here, $\sigma = \sigma(x, y, z)$ is specific heat capacity (`[J/(m`^{3}`K)]`). The second relation is Fourier's law, which states that the heat flux is proprtional to the temperature gradient:

$$\mathbf{j} = -\lambda \operatorname{grad} u = -\lambda \nabla u, \tag{3}$$

where $\lambda = \lambda(x, y, z)$ is the heat conductivity `[J/(mKs)]`. Substituting, (2) and (3) into (1) we obtain the **heat equation**:

$$\sigma \partial_t u - \nabla \cdot (\lambda \nabla u) = p \quad \text{in } \Omega, \text{ for } t > 0. \tag{4}$$

1.1.1 Special cases: stationary heat equation

When the temperutre in the solid is in equilibrium; that is, when $\partial_t u$, then we obtain the **stationary heat equation**

$$-\nabla \cdot (\lambda \nabla u) = p \quad \text{in } \Omega.$$

If λ is constant, then

$$-\nabla \cdot (\lambda \nabla u) = -\lambda \nabla \cdot \nabla u = -\lambda \Delta u,$$

where

$$\Delta u = \nabla \cdot \nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

is called the **Laplace operator**. Hence, in this case, we get (with $f = -p/\lambda$),

$$\Delta u = f \text{ in } \Omega,$$

which is called **Poisson's equation**. When $f = 0$, this reads

$$\Delta u = 0 \text{ in } \Omega,$$

which is called **Laplace's equation**.

1.1.2 Boundary conditions

In order to supplement the heat equation with boundary conditions, we assume that the heat flux through the boundary Γ is proportional to the difference of the temperature of the surface of the solid and the ambient temperature T_A, reduced by a possibly a prescribed heat influx (for example, through heating) $g = g(x, y, z, t)$ ($[\mathtt{J/(m^2s)}]$):

$$\mathbf{j} \cdot \mathbf{n} = \kappa(T - T_A) - g = \kappa(u - u_A) - g, \text{ on } \Gamma \quad u_A = T_A - T_0, \tag{5}$$

where $\kappa = \kappa(x, y, z)$ is the heat transfer coefficient ($[\mathtt{J/(m^2sK)}]$). The heat flux should also obey Fourier's law at the boundary:

$$\mathbf{j} \cdot \mathbf{n} = -\lambda \nabla u \cdot n. \tag{6}$$

Introducing the notation

$$D_N u = \nabla u \cdot n$$

and equating (5) and (6), one arrives at

$$\lambda D_N u + \kappa(u - u_A) = g \text{ on } \Gamma. \tag{7}$$

Special cases:
1. Perfect isolation: $\kappa = 0$. Then, (7) becomes

$$\lambda D_N u = g \text{ on } \Gamma,$$

which is called a **Neumann boundary condition**.
2. No isolation: $\kappa = \infty$. One divides (7) by κ,

$$\frac{1}{\kappa} \lambda D_N u + (u - u_A) = \frac{1}{\kappa} g \text{ on } \Gamma,$$

and let $\kappa \to \infty$. We get $u - u_A = 0$ on Γ or

$$u = u_A \text{ on } \Gamma.$$

This is called a **Dirichlet boundary condition.**

1.2 Boundary value problem and weak formulation

Let Ω be a bounded solid in $\mathbf{R}^3$ with piecewise smooth positively oriented (= outward normal) boundary surface Γ.

The **boundary value problem** is: find $u = u(x, y, z)$ such that

$$\begin{cases} -\nabla \cdot (\lambda \nabla u) = p & \text{in } \Omega, \\ \lambda\, \mathrm{D}_N u + \kappa(u - u_{\mathrm{A}}) = g & \text{on } \Gamma. \end{cases} \tag{8}$$

In order to derive the weak formulation of this problem, one needs an integration by parts formula in 3 dimensions. Let ϕ be a continuously differentiable scalar field and $\mathbf{F}$ be a continuously differentiable vector field. Then one has the product rule

$$\operatorname{div}(\phi \mathbf{F}) = \nabla \cdot (\phi F) = \mathbf{F} \cdot \nabla \phi + \phi \nabla \cdot \mathbf{F} = \mathbf{F} \cdot \operatorname{grad} \phi + \phi \operatorname{div} \mathbf{F}. \tag{9}$$

This can be proved in a straightforward fashion by writing out the definitions of div and grad and using the one dimensional product rule (See, problem 25 in Chapter 16.6 of Stewart). Integration over Ω gives

$$\iiint\limits_{\Omega} \operatorname{div}(\phi \mathbf{F})\, \mathrm{d}V = \iiint\limits_{\Omega} \mathbf{F} \cdot \operatorname{grad} \phi\, \mathrm{d}V + \iiint\limits_{\Omega} \phi \operatorname{div} \mathbf{F}\, \mathrm{d}V.$$

Using the Divergence Theorem, this yields

$$\iint\limits_{\Gamma} \phi\mathbf{F}\cdot\,\mathrm{d}\mathbf{S} = \iiint\limits_{\Omega} \mathbf{F}\cdot\operatorname{grad}\phi\,\mathrm{d}V + \iiint\limits_{\Omega} \phi\operatorname{div}\mathbf{F}\,\mathrm{d}V.$$

We rearrange, use the ∇ notation and the definition

$$\iint\limits_{\Gamma} \phi\mathbf{F}\cdot\,\mathrm{d}\mathbf{S} = \iint\limits_{\Gamma} \phi\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S$$

to arrive at the integration by parts formula

$$\iiint\limits_{\Omega} \phi\nabla\cdot\mathbf{F}\,\mathrm{d}V = \iint\limits_{\Gamma} \phi\mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S - \iiint\limits_{\Omega} \mathbf{F}\cdot\nabla\phi\,\mathrm{d}V. \tag{10}$$

Now we consider the heat equation in (8). We multiply the first equation in (8) with a test function $v = v(x, y, z)$, integrate over the domain Ω and use the integration by parts formula (10) with $\mathbf{F} = \lambda\nabla u$ and $\phi = v$:

$$\iiint\limits_{\Omega} pv\,\mathrm{d}V = -\iiint\limits_{\Omega} v\nabla\cdot(\lambda\nabla u)\,\mathrm{d}V = -\iint\limits_{\Gamma} v\lambda\nabla u\cdot\mathbf{n}\,\mathrm{d}S + \iiint\limits_{\Omega} \lambda\nabla u\cdot\nabla v\,\mathrm{d}V. \tag{11}$$

We use the boundary condition, the second equation in (8), to write

$$\lambda\nabla u\cdot\mathbf{n} = \lambda D_N u = g - \kappa(u - u_A).$$

Inserting this to (11) we obtain

$$\iiint\limits_{\Omega} pv\,\mathrm{d}V = \iint\limits_{\Gamma} \kappa uv\,\mathrm{d}S - \iint\limits_{\Gamma} (g + \kappa u_A)v\,\mathrm{d}S + \iiint\limits_{\Omega} \lambda\nabla u\cdot\nabla v\,\mathrm{d}V.$$

Hence, the **weak formulation** of (8) reads:

Find $u = u(x, y, z)$ such that

$$\iiint\limits_{\Omega} \lambda\nabla u\cdot\nabla v\,\mathrm{d}V + \iint\limits_{\Gamma} \kappa uv\,\mathrm{d}S = \iiint\limits_{\Omega} pv\,\mathrm{d}V + \iint\limits_{\Gamma} (g + \kappa u_A)v\,\mathrm{d}S \tag{12}$$

for every test function v.

As in the one dimensional case, for the precise mathematical formulation one would have to specify the exact function spaces to which u and v belongs to. This is beyond the scope of this course.

Note. Often different boundary conditions are specified on different parts of the boundary Γ. In this case, the weak formulation changes. For example, consider the boundary value problem:

Find $u = u(x, y, z)$ such that

$$\begin{cases} -\nabla\cdot(\lambda\nabla u) = p & \text{in } \Omega, \\ u = u_B & \text{on } \Gamma_1, \\ \lambda\,\mathrm{D}_N u + \kappa(u - u_{\mathrm{A}}) = g & \text{on } \Gamma_2, \end{cases} \tag{13}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$ and Γ_1 and Γ_2 are disjoint except at the curve where they meet. Note that on Γ_1 we prescribed a Dirichlet boundary condition which is special and this has to be taken into account appropriately in the weak formulation as follows:

Find $u = u(x,y,z)$ such that $u = u_B$ on Γ_1 and

$$\iiint_{\Omega} \lambda \nabla u \cdot \nabla v \,\mathrm{d}V + \iint_{\Gamma_2} \kappa u v \,\mathrm{d}S = \iiint_{\Omega} p v \,\mathrm{d}V + \iint_{\Gamma_2} (g + \kappa u_A) v \,\mathrm{d}S \tag{14}$$

for every test function v such that $v = 0$ on Γ_1.

In particular, when $\Gamma_2 = \emptyset$; that is when $u = u_B$ on the whole of $\Gamma = \Gamma_1$, then both boundary integral terms in (14) disappear completely.

1.3 The stationary heat equation and FEM in 2D

In this section we consider the stationary heat equation in 2D and its finite element approximation. Let now Ω be a bounded planar domain and Γ be its piecewise smooth boundary with positive (counterclockwise) orientation. Then the **boundary value problem** we consider reads as follows:

Find $u = u(x,y)$ such that

$$\begin{cases} -\nabla \cdot (\lambda \nabla u) = f & \text{in } \Omega, \\ \lambda\, \mathrm{D}_N u + \kappa (u - u_{\mathrm{A}}) = g & \text{on } \Gamma. \end{cases} \tag{15}$$

In order to derive the weak formulation of (15), we need a 2D version of integration by parts. Recall Green's theorem which states that if $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is a continuously differentiable vector field in 2D and Ω is a bounded planar domain with piecewise smooth boundary Γ with positive (counterclockwise) orientation, then

$$\iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A = \int_{\Gamma} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\Gamma} \mathbf{F} \cdot \mathbf{r_0} \,\mathrm{d}s,$$

where $\mathbf{r_0}$ is the unit tangent vector of Γ. Using this one can derive the following form of Green's theorem, see Stewart, Section 16.5 page 1097, formula 13:

$$\iint_{\Omega} \operatorname{div} \mathbf{F} \,\mathrm{d}A = \iint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \mathrm{d}A = \int_{\Gamma} \mathbf{F} \cdot \mathbf{n} \,\mathrm{d}s, \tag{16}$$

where $\mathbf{n}$ is the outward pointing unit normal vector to Γ. By integrating the product rule (9) over Ω we get

$$\iint_{\Omega} \operatorname{div} (\phi \mathbf{F}) \,\mathrm{d}A = \iint_{\Omega} \mathbf{F} \cdot \operatorname{grad} \phi \,\mathrm{d}A + \iint_{\Omega} \phi \operatorname{div} \mathbf{F} \,\mathrm{d}A.$$

Using (16), this yields

$$\int_{\Gamma} \phi \mathbf{F} \cdot \mathbf{n} \,\mathrm{d}s = \iint_{\Omega} \mathbf{F} \cdot \operatorname{grad} \phi \,\mathrm{d}A + \iint_{\Omega} \phi \operatorname{div} \mathbf{F} \,\mathrm{d}A.$$

We rearrange and use the ∇ notation to arrive at the integration by parts formula in 2D:

$$\iint_{\Omega} \phi \nabla \cdot \mathbf{F} \,\mathrm{d}A = \int_{\Gamma} \phi \mathbf{F} \cdot \mathbf{n} \,\mathrm{d}s - \iint_{\Omega} \mathbf{F} \cdot \nabla \phi \,\mathrm{d}A. \tag{17}$$

Now we consider the heat equation in (15). We multiply the first equation in (15) with a test function $v = v(x,y)$, integrate over the domain Ω and use the integration by parts formula (17) with $\mathbf{F} = \lambda \nabla u$ and $\phi = v$:

$$\iint_{\Omega} f v \,\mathrm{d}A = -\iint_{\Omega} v \nabla \cdot (\lambda \nabla u) \,\mathrm{d}A = -\int_{\Gamma} v \lambda \nabla u \cdot \mathbf{n} \,\mathrm{d}s + \iint_{\Omega} \lambda \nabla u \cdot \nabla v \,\mathrm{d}A. \tag{18}$$

We use the boundary condition, the second equation in (15), to write

$$\lambda \nabla u \cdot \mathbf{n} = \lambda D_N u = g - \kappa(u - u_A).$$

Inserting this to (18) we obtain

$$\iint\limits_{\Omega} f v \,\mathrm{d}A = \int\limits_{\Gamma} \kappa u v \,\mathrm{d}s - \int\limits_{\Gamma} (g + \kappa u_A) v \,\mathrm{d}s + \iint\limits_{\Omega} \lambda \nabla u \cdot \nabla v \,\mathrm{d}A.$$

Hence, the **weak formulation** of (15) reads:

Find $u = u(x, y)$ such that

$$\iint\limits_{\Omega} \lambda \nabla u \cdot \nabla v \,\mathrm{d}A + \int\limits_{\Gamma} \kappa u v \,\mathrm{d}s = \iint\limits_{\Omega} f v \,\mathrm{d}A + \int\limits_{\Gamma} (g + \kappa u_A) v \,\mathrm{d}s \tag{19}$$

for every test function v.

Note. Similarly to the 3D case, often different boundary conditions are specified on different parts of the boundary curve Γ. In this case the weak formulation changes. For example, consider the boundary value problem in 2D:

Find $u = u(x, y)$ such that

$$\begin{cases} -\nabla \cdot (\lambda \nabla u) = f & \text{in } \Omega, \\ u = u_B & \text{on } \Gamma_1, \\ \lambda\, \mathrm{D}_N u + \kappa(u - u_{\mathrm{A}}) = g & \text{on } \Gamma_2, \end{cases} \tag{20}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$ and Γ_1 and Γ_2 are disjoint except at the points where they meet. Note that on Γ_1 we prescribed a Dirichlet boundary condition which is special and this has to be taken into account appropriately in the weak formulation as follows:

Find $u = u(x, y)$ such that $u = u_B$ on Γ_1 and

$$\iint\limits_{\Omega} \lambda \nabla u \cdot \nabla v \,\mathrm{d}A + \int\limits_{\Gamma_2} \kappa u v \,\mathrm{d}s = \iint\limits_{\Omega} f v \,\mathrm{d}A + \int\limits_{\Gamma_2} (g + \kappa u_A) v \,\mathrm{d}s \tag{21}$$

for every test function v such that $v = 0$ on Γ_1.

In particular, when $\Gamma_2 = \emptyset$; that is when $u = u_B$ on the whole of $\Gamma = \Gamma_1$, then both boundary integral terms in (21) disappear completely.

1.3.1 FEM in 2D

Let Ω be a polygonal domain, for simplicity, and consider a triangulation (triangular mesh) of Ω. A mesh constists of

$$\begin{aligned} &N \text{ nodes } \{P_i\}_{i=1}^N, \\ &M \text{ triangles } \{T_j\}_{j=1}^M, \\ &L \text{ edges } \{E_l\}_{l=1}^L. \end{aligned}$$

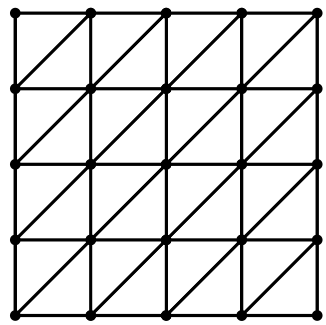

Figure 2: Triangulation of a rectangular planar domain Ω.

A *continuous piecewise linear* function U is a continuous function on Ω such that $U(x, y) = a + bx + cy$ (plane) on every triangle T_j (of course, the constants a, b, c usually change from tringle to triangle). As 3 points in space determines a plane such a function U is completely determined by its nodal values $U(P_i)$:

$$U(x, y) = \sum_{i=1}^{N} U_i \phi_i(x, y), \quad U_i = U(P_i). \tag{22}$$

Here, $\{\phi_i\}_{i=1}^{N}$ are the basis functions, defined to be continuous piecewise linear functions such that

$$\phi_i(P_j) = \begin{cases} 1, & \text{om } i = j, \\ 0, & \text{om } i \neq j. \end{cases}$$

These are also called pyramid functions (see Figure (3) for a typical example.)

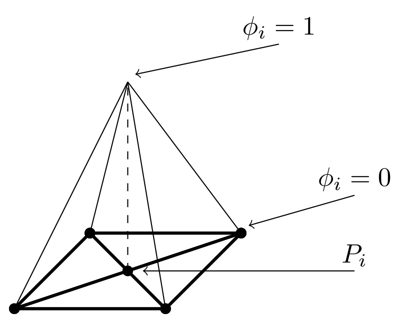

Figure 3: Basis function (”pyramid function”).

We look for an approximation of the solution u of (15) of the form $U(x, y) = \sum_{i=1}^{N} U_i \phi_i(x, y)$ and hence we need to determine the nodal values U_i of U. As continuous piecewise linear functions do not have two derivatives, we use the weak formulation (19) instead of the original formulation (15). We replace u by U in (19) and use the special choice $v = \phi_j$ as test functions:

$$\iint_{\Omega} \lambda \nabla \left(\sum_{i=1}^{N} U_i \phi_i \right) \cdot \nabla \phi_j \, \mathrm{d}A + \int_{\Gamma} \kappa \left(\sum_{i=1}^{N} U_i \phi_i \right) \phi_j \, \mathrm{d}s$$
$$= \iint_{\Omega} f \phi_j \, \mathrm{d}A + \int_{\Gamma} (g + \kappa u_A) \phi_j \, \mathrm{d}s, \quad j = 1, \ldots, N.$$

Factoring out the coefficients U_i we get:

$$\sum_{i=1}^{N} U_i \iint\limits_{\Omega} \lambda \nabla\phi_i \cdot \nabla\phi_j \,\mathrm{d}A + \sum_{i=1}^{N} U_i \int\limits_{\Gamma} \kappa\phi_i\phi_j \,\mathrm{d}s$$
$$= \iint\limits_{\Omega} f\phi_j \,\mathrm{d}A + \int\limits_{\Gamma} (g+\kappa u_A)\phi_j \,\mathrm{d}s, \quad j = 1,\ldots,N.$$

or, after collecting terms,

$$\sum_{i=1}^{N} U_i \underbrace{\left(\iint\limits_{\Omega} \lambda \nabla\phi_i \cdot \nabla\phi_j \,\mathrm{d}A + \int\limits_{\Gamma} \kappa\phi_i\phi_j \,\mathrm{d}s \right)}_{=a_{ji}}$$
$$= \underbrace{\iint\limits_{\Omega} f\phi_j \,\mathrm{d}A + \int\limits_{\Gamma} (g+\kappa u_A)\phi_j \,\mathrm{d}s,}_{=b_j} \quad j = 1,\ldots,N.$$

This is of the form

$$\sum_{i=1}^{N} a_{ji} U_i = b_j, \quad j = 1,\ldots,N;$$

that is, a linear system of equations for U_i. We rewrite this in the matrix form as

$$\mathcal{A}\mathcal{U} = b,$$

with

$$\mathcal{U} = \begin{bmatrix} U_1 \\ \vdots \\ U_N \end{bmatrix}$$

and stiffness matrix

$$\mathcal{A} = \begin{bmatrix} a_{11} & \ldots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \ldots & a_{NN} \end{bmatrix}, \quad a_{ji} = \iint\limits_{\Omega} \lambda \nabla\phi_i \cdot \nabla\phi_j \,\mathrm{d}A + \int\limits_{\Gamma} \kappa\phi_i\phi_j \,\mathrm{d}s$$

and load vector

$$b = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}, \quad b_j = \iint\limits_{\Omega} f\phi_j \,\mathrm{d}A + \int\limits_{\Gamma} (g+\kappa u_A)\phi_j \,\mathrm{d}s.$$

The matrix $\mathcal{A}$ is *symmetric*: $A = A^T$ ($a_{ij} = a_{ji}$), and is usually very *large*: the number N of nodes is large (for example, $N = 10^4$ or more.) However, the matrix $\mathcal{A}$ is *sparse*: for most matrix elements we have $a_{ij} = 0$. We only have $a_{ij} \neq 0$ when the corresponding nodes P_i and P_j are neighbours.

PDE Toolbox

The MATLAB-program PDE Toolbox sets up the linear system of equations $\mathcal{A}\mathcal{U} = b$ and solves it.

1.4 The time dependent heat equation

We consider the 2D version of the time dependent heat equation (4). Let Ω be a bounded planar domain and Γ be its piecewise smooth boundary with positive (counterclockwise) orientation. Then the **initial-boundary value problem** we consider reads as follows:

Find $u = u(x, y, t)$ such that

$$
(23) \qquad \begin{cases} \partial_t u(x,y,t) - \nabla \cdot (\lambda(x,y)\nabla u(x,y,t)) = f(x,y,t) & (x,y) \in \Omega, \quad t > 0, \\ \lambda\, \mathrm{D}_N u(x,y,t) + \kappa(x,y)(u(x,y) - u_{\mathrm{A}}(t)) = g(x,y,t) & (x,y) \in \Gamma, \quad t > 0, \\ u(x,y,0) = w(x,y) & (x,y) \in \Omega. \end{cases}
$$

1.4.1 Weak formulation

We derive the weak formulation the same way as for the stationary heat equation by multiplying the first equation in (23) by a test function $v = v(x, y)$, integrate over the domain Ω and use the boundary condition, the second equation in (23) after integrating by parts. The weak formulation of (23) then becomes:

Find $u = u(x, y, t)$ such that $u(x, y, 0) = w(x, y)$ and for $t > 0$,

$$
(24) \qquad \iint_\Omega \partial_t uv \,\mathrm{d}A + \iint_\Omega \lambda \nabla u \cdot \nabla v \,\mathrm{d}A + \int_\Gamma \kappa uv \,\mathrm{d}s = \iint_\Omega fv \,\mathrm{d}A + \int_\Gamma (g + \kappa u_A)v \,\mathrm{d}s
$$

for every test function v.

The novelty in this weak formulation compared to the stationary case is the requirement that $u(x, y, 0) = w(x, y)$ and the appearance of the term $\iint_\Omega \partial_t uv \,\mathrm{d}A$ on the left hand side of (24) which is not present in the stationary case (19).

1.4.2 FEM

As in the stationary case, let Ω be a polygonal domain, for simplicity, and consider a triangulation of Ω with nodes P_i, $i = 1, \dots, N$. We look for an approximation of u in the form $U(x, y, t) = \sum_{i=1}^N U_i(t)\phi_i(x, y)$, where ϕ_i is the finite element basis function corresponding to P_i. We need to determine the nodal values $U_i(t)$ of U. As in the stationary case we replace u by U in the weak formulation (24) and use the test functions $v = \phi_j$, $j = 1, \dots, N$, to get

$$
\sum_{i=1}^N \dot{U}_i(t) \iint_\Omega \phi_i\phi_j \mathrm{d}A + \sum_{i=1}^N U_i \iint_\Omega \lambda\nabla\phi_i \cdot \nabla\phi_j \,\mathrm{d}A + \sum_{i=1}^N U_i \int_\Gamma \kappa\phi_i\phi_j \,\mathrm{d}s
$$
$$
= \iint_\Omega f\phi_j \,\mathrm{d}A + \int_\Gamma (g + \kappa u_A)\phi_j \,\mathrm{d}s, \quad j = 1, \dots, N.
$$

or, after collecting terms,

$$
\sum_{i=1}^N \dot{U}_i(t) \underbrace{\iint_\Omega \phi_i\phi_j \mathrm{d}A}_{=m_j i} + \sum_{i=1}^N U_i \underbrace{\left(\iint_\Omega \lambda\nabla\phi_i \cdot \nabla\phi_j \,\mathrm{d}A + \int_\Gamma \kappa\phi_i\phi_j \,\mathrm{d}s \right)}_{=a_{ji}}
$$
$$
= \underbrace{\iint_\Omega f\phi_j \,\mathrm{d}A + \int_\Gamma (g + \kappa u_A)\phi_j \,\mathrm{d}s}_{=b_j(t)}, \quad j = 1, \dots, N.
$$

This is of the form

$$\sum_{i=1}^{N} m_{ji}\dot{U}_i(t) + \sum_{i=1}^{N} a_{ji}U_i(t) = b_j(t), \quad j = 1, \dots, N;$$

that is, a linear system of differential equations for U_i. We rewrite this in the matrix form as

$$\mathcal{M}\dot{\mathcal{U}}(t) + \mathcal{A}\mathcal{U}(t) = b(t), \quad t > 0, \tag{25}$$

with

$$\mathcal{U}(t) = \begin{bmatrix} U_1(t) \\ \vdots \\ U_N(t) \end{bmatrix}, \quad \dot{\mathcal{U}}(t) = \begin{bmatrix} \dot{U}_1(t) \\ \vdots \\ \dot{U}_N(t) \end{bmatrix},$$

stiffness matrix

$$\mathcal{A} = \begin{bmatrix} a_{11} & \dots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \dots & a_{NN} \end{bmatrix}, \quad a_{ij} = a_{ji} = \iint_{\Omega} \lambda \nabla\phi_i \cdot \nabla\phi_j \, \mathrm{d}A + \int_{\Gamma} \kappa\phi_i\phi_j \, \mathrm{d}s,$$

mass matrix

$$\mathcal{M} = \begin{bmatrix} m_{11} & \dots & m_{1N} \\ \vdots & \ddots & \vdots \\ m_{N1} & \dots & m_{NN} \end{bmatrix}, \quad m_{ij} = m_{ji} = \iint_{\Omega} \phi_i\phi_j \mathrm{d}A,$$

and load vector

$$b(t) = \begin{bmatrix} b_1(t) \\ \vdots \\ b_N(t) \end{bmatrix}, \quad b_j(t) = \iint_{\Omega} f(x, y, t)\phi_j(x, y) \, \mathrm{d}A + \int_{\Gamma} (g(x, y, t) + \kappa(x, y)u_A(t))\phi_j(x, y) \, \mathrm{d}s.$$

In order to determine $U_1(t), \dots, U_N(t)$ one needs to solve (25) which is a linear, first order differential equation system that could be solved approximately by a time-stepping method, such as, the Backward Euler method. It requires an initial vector

$$\mathcal{U}(0) = \begin{bmatrix} U_1(0) \\ \vdots \\ U_N(0) \end{bmatrix}, \text{ with } U_j(0) = \iint_{\Omega} w\phi_j \mathrm{d}A, \quad j = 1, \dots, N,$$

where w is the initial condition from (23).

1.5 The wave equation in 2D

Here we consider the wave equation that can be used, for example, to describe the displacement u of a vibrating plate of the shape of Ω. Let Ω be a bounded planar domain and Γ be its piecewise smooth boundary with positive (counterclockwise) orientation. Then the **initial-boundary value problem** we consider reads as follows:

Find $u = u(x, y, t)$ such that

$$\begin{cases} \partial_t^2 u(x, y, t) - (a(x, y))^2 \Delta u(x, y, t) = f(x, y, t) & (x, y) \in \Omega, \quad t > 0, \\ \tau(x, y) \, \mathrm{D}_N u(x, y, t) + k(x, y)u(x, y, t) = g(x, y, t) & (x, y) \in \Gamma, \quad t > 0, \\ u(x, y, 0) = w_1(x, y) & (x, y) \in \Omega, \\ \partial_t u(x, y, 0) = w_2(x, y) & (x, y) \in \Omega. \end{cases} \tag{26}$$

Here $\Delta u = \nabla \cdot (\nabla u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. As the the equation contains two time derivatives of u one needs two initial conditions, one for u and one for $\partial_t u$.

1.5.1 Weak formulation

For simplicity we take $a(x,y) = \tau(x,y) = 1$ to be constant. Then the initial-boundary value problem (26) simplifies to

$$
\text{(27)} \qquad \begin{cases} \partial_t^2 u(x,y,t) - \Delta u(x,y,t) = f(x,y,t) & (x,y) \in \Omega, \quad t > 0, \\ \mathrm{D}_N u(x,y,t) + k(x,y)u(x,y,t) = g(x,y,t) & (x,y) \in \Gamma, \quad t > 0, \\ u(x,y,0) = w_1(x,y) & (x,y) \in \Omega, \\ \partial_t u(x,y,0) = w_2(x,y) & (x,y) \in \Omega. \end{cases}
$$

To derive the weak formulation of (27) we multiply the wave equation, the first equation in (27), by a test function $v = v(x,y)$, integrate over the domain Ω and use the integration by parts formula (17) with $\mathbf{F} = \nabla u$ and $\phi = v$:

$$
\begin{aligned}
\text{(28)} \quad \iint\limits_{\Omega} fv \,\mathrm{d}A = \iint\limits_{\Omega} \partial_t^2 uv \,\mathrm{d}A - \iint\limits_{\Omega} \Delta uv \,\mathrm{d}A &= \iint\limits_{\Omega} \partial_t^2 uv \,\mathrm{d}A - \iint\limits_{\Omega} v\nabla \cdot (\nabla u) \,\mathrm{d}A \\
&= \iint\limits_{\Omega} \partial_t^2 uv \,\mathrm{d}A - \int\limits_{\Gamma} v\nabla u \cdot \mathbf{n} \,\mathrm{d}s + \iint\limits_{\Omega} \nabla u \cdot \nabla v \,\mathrm{d}A.
\end{aligned}
$$

We use the boundary condition, the second equation in (27), to write

$$
\nabla u \cdot \mathbf{n} = D_N u = g - ku.
$$

Inserting this into (28) we obtain

$$
\iint\limits_{\Omega} fv \,\mathrm{d}A = \iint\limits_{\Omega} \partial_t^2 uv \,\mathrm{d}A + \int\limits_{\Gamma} kuv \,\mathrm{d}s - \int\limits_{\Gamma} gv \,\mathrm{d}s + \iint\limits_{\Omega} \nabla u \cdot \nabla v \,\mathrm{d}A.
$$

Therefore the **weak formulation** of (27) reads as:

Find $u = u(x,y,t)$ such that $u(x,y,0) = w_1(x,y)$, $\partial_t u(x,y,0) = w_2(x,y)$, and for $t > 0$,

$$
\text{(29)} \qquad \iint\limits_{\Omega} \partial_t^2 uv \,\mathrm{d}A + \iint\limits_{\Omega} \nabla u \cdot \nabla v \,\mathrm{d}A + \int\limits_{\Gamma} kuv \,\mathrm{d}s = \iint\limits_{\Omega} fv \,\mathrm{d}A + \int\limits_{\Gamma} gv \,\mathrm{d}s,
$$

for every test function v.

1.5.2 FEM

As before, for simplicity, let Ω be a polygonal domain, and consider a triangulation of Ω with nodes P_i, $i = 1, \ldots, N$. We look for an approximation of u in the form $U(x,y,t) = \sum_{i=1}^{N} U_i(t)\phi_i(x,y)$, where ϕ_i is the finite element basis function corresponding to P_i. We need to determine the nodal values $U_i(t)$ of U. As before, we replace u by U in the weak formulation (29) and use the test functions $v = \phi_j$, $j = 1, \ldots, N$, to get

$$
\begin{aligned}
\sum_{i=1}^{N} \ddot{U}_i(t) \iint\limits_{\Omega} \phi_i\phi_j \mathrm{d}A + \sum_{i=1}^{N} U_i \iint\limits_{\Omega} \nabla\phi_i \cdot \nabla\phi_j \,\mathrm{d}A + \sum_{i=1}^{N} U_i \int\limits_{\Gamma} k\phi_i\phi_j \,\mathrm{d}s& \\
= \iint\limits_{\Omega} f\phi_j \,\mathrm{d}A + \int\limits_{\Gamma} g\phi_j \,\mathrm{d}s, \quad j = 1, \ldots, N.&
\end{aligned}
$$

or, after collecting terms,

$$\sum_{i=1}^{N} \ddot{U}_i(t) \underbrace{\iint\limits_{\Omega} \phi_i \phi_j \mathrm{d}A}_{=m_j i} + \sum_{i=1}^{N} U_i \underbrace{\left(\iint\limits_{\Omega} \nabla \phi_i \cdot \nabla \phi_j \,\mathrm{d}A + \int\limits_{\Gamma} k \phi_i \phi_j \,\mathrm{d}s \right)}_{=a_{ji}}$$

$$= \underbrace{\iint\limits_{\Omega} f \phi_j \,\mathrm{d}A + \int\limits_{\Gamma} g \phi_j \,\mathrm{d}s}_{=b_j(t)}, \quad j = 1, \ldots, N.$$

This is of the form

$$\sum_{i=1}^{N} m_{ji} \ddot{U}_i(t) + \sum_{i=1}^{N} a_{ji} U_i(t) = b_j(t), \quad j = 1, \ldots, N;$$

that is, a linear system of differential equations for U_i. We rewrite this in the matrix form as

$$\mathcal{M}\ddot{\mathcal{U}}(t) + \mathcal{A}\mathcal{U}(t) = b(t), \quad t > 0, \tag{30}$$

with

$$\mathcal{U}(t) = \begin{bmatrix} U_1(t) \\ \vdots \\ U_N(t) \end{bmatrix}, \quad \ddot{\mathcal{U}}(t) = \begin{bmatrix} \ddot{U}_1(t) \\ \vdots \\ \ddot{U}_N(t) \end{bmatrix},$$

stiffness matrix

$$\mathcal{A} = \begin{bmatrix} a_{11} & \ldots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \ldots & a_{NN} \end{bmatrix}, \quad a_{ij} = a_{ji} = \iint\limits_{\Omega} \nabla \phi_i \cdot \nabla \phi_j \,\mathrm{d}A + \int\limits_{\Gamma} k \phi_i \phi_j \,\mathrm{d}s,$$

mass matrix

$$\mathcal{M} = \begin{bmatrix} m_{11} & \ldots & m_{1N} \\ \vdots & \ddots & \vdots \\ m_{N1} & \ldots & m_{NN} \end{bmatrix}, \quad m_{ij} = m_{ji} = \iint\limits_{\Omega} \phi_i \phi_j \mathrm{d}A,$$

and load vector

$$b(t) = \begin{bmatrix} b_1(t) \\ \vdots \\ b_N(t) \end{bmatrix}, \quad b_j(t) = \iint\limits_{\Omega} f(x, y, t) \phi_j(x, y) \,\mathrm{d}A + \int\limits_{\Gamma} g(x, y, t) \phi_j(x, y) \,\mathrm{d}s.$$

In order to determine $U_1(t), \ldots, U_N(t)$ one needs to solve (30) which is a linear, second order differential equation system that could be solved approximately by a time-stepping method, such as, the Backward Euler method, after rewriting it as a first order system. It requires two initial vectors

$$\mathcal{U}(0) = \begin{bmatrix} U_1(0) \\ \vdots \\ U_N(0) \end{bmatrix}, \text{ with } U_j(0) = \iint\limits_{\Omega} w_1 \phi_j \mathrm{d}A, \quad j = 1, \ldots, N,$$

where w_1 is the first initial condition from (27) and

$$\dot{\mathcal{U}}(0) = \begin{bmatrix} \dot{U}_1(0) \\ \vdots \\ \dot{U}_N(0) \end{bmatrix}, \text{ with } \dot{U}_j(0) = \iint\limits_{\Omega} w_2 \phi_j \mathrm{d}A, \quad j = 1, \ldots, N,$$

where w_2 is the second initial condition from (27).