Proposition 1.1. *With the inner product defined by* $(x, y) := \sum_{i=1}^{\infty} x_i y_i$*,* ℓ^2 *is a Hilbert space.*

Proof. We only have to prove that ℓ^2 is complete. Suppose $\{x_n\}$ is a Cauchy sequence in ℓ^2, and write $x_n = (x_{n,1}, x_{n,2}, \cdots)$. Then for each fixed j,

$$|x_{m,j} - x_{n,j}|^2 \leq \sum_{i=1}^{\infty} |x_{m,i} - x_{n,i}|^2 = \|x_m - x_n\|^2,$$

so $\{x_{n,j}\}$ is Cauchy and converges to y_j, say. We want to prove that $y \in \ell^2$ and $\|x_n - y\| \to 0$.

Fix $\epsilon > 0$, choose N such that

$$m, n \geq N \Longrightarrow \|x_m - x_n\| < \epsilon, \tag{1.1}$$

and fix $m \geq N$; we aim to prove that $\|x_m - y\| \leq \epsilon$. (If you are nervous about the $\leq \epsilon$ here, replace the ϵ in (1.1) by $\epsilon/2$; then we get $\leq \epsilon/2 < \epsilon$ at the end.) We know from (1.1) that for every $n \geq N$ and every $K \in \mathbb{N}$ we have

$$\sum_{i=1}^{K} |x_{m,i} - x_{n,i}|^2 \leq \sum_{i=1}^{\infty} |x_{m,i} - x_{n,i}|^2 = \|x_m - x_n\|^2 < \epsilon^2;$$

since there are only finitely many summands and $x_{n,i} \to y_i$ for each i, we see that

$$\sum_{i=1}^{K} |x_{m,i} - x_{n,i}|^2 \to \sum_{i=1}^{K} |x_{m,i} - y_i|^2 \text{ as } n \to \infty.$$

Thus

$$\sum_{i=1}^{K} |x_{m,i} - y_i|^2 \leq \epsilon^2$$

for every K. But this in turn implies that

$$\sum_{i=1}^{\infty} |x_{m,i} - y_i|^2 \leq \epsilon^2.$$

This says both that $x_m - y$ belongs to ℓ^2, so $y = x_m - (x_m - y)$ is a linear combination of elements of ℓ^2, hence belongs to ℓ^2, and that $\|x_m - y\| \leq \epsilon$, as required. □