TMA690 - Project description

Maximilian Thaller

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1 General description

1.1 Goals

The goal of the project is to give a first experience in the implementation of the Finite Element Method (FEM). The students will see how physical problems can be described by boundary value problems consisting in a system of partial differential equations satisfied on a domain Ω and equipped with boundary conditions. They will learn how to set up the weak formulation of a boundary value problem, how to correctly handle boundary conditions, and how to discretize a domain. Eventually the discretized problem will be solved and the students shall experience how different parameter choices in the boundary value problem reflect different situations in the underlying physical problem.

Moreover, besides the mathematical skills, it should be trained how to work on a project more or less autonomously and how to properly present the results in a report. It is important to give a correct and complete description of the mathematical and numerical tools that are used in the project. Finally, the results have to be documented in detail and presented in a way such that they can be reproduced by the reader.

1.2 General guidelines

The following guidelines have to be respected when carrying out the projects:

- The projects should be carried out in groups of two. The students form the groups autonomously. If there are problems, e.g. someone does not find a partner, a solution will be found together with the supervisor. Groups of one or three are okay as exceptions.
- Each team chooses one of the projects described in Section 2 and works on it according to the exercises stated. The project is mandatory and the students can receive up to five bonus points for the exam.
- The teams have the possibility to propose an own project to the supervisor that can be carried out instead of the projects described in this text.

- The result of the project will be a report containing

  1. a description of the physical problem and how it is modeled by a boundary value problem,
  2. the weak formulation of the boundary value problem,
  3. the process of discretization of the problem,
  4. an explanation of the numerical methods and the code,
  5. a section where the code is tested and the boundary value problem is solved containing some graphs,
  6. and finally the Matlab - source code (equipped with comments).

- Deadline for handing in the report is **Friday, December 21ST, 23:59**. The reports should be sent to the supervisor via e-mail.

- The reports can be written in Swedish, English, French or German.

- The students get help with their projects in meetings with the supervisor. They are very much encouraged to book at least one meeting with the supervisor.

- For the first meeting the students should have formed a team, selected a project and written down the weak formulation.

- The evaluation criteria are:

  - Report containing the points (1) - (6) from above with no severe mistakes: passed, 0 bonus points
  - Entirely correct and complete presentation of the theoretical part: 1 bonus point.
  - Professional presentation of the result with proper images and discussion of errors: 1 bonus point.
  - One obtains further bonus points by completing the according tasks in the project descriptions in Section 2.
  - From the 3rd meeting with the supervisor onwards: -1 bonus point per meeting.

2 Projects

2.1 Heat conduction in a water hose

In this project we wish to simulate heat conduction in a non insulated water hose filled with water that is assumed to be at rest. The surrounding of the hose has a temperature of 300K. We assume that the water in the hose is hit by microwaves that cause the water to heat up. The microwaves act as heat sources because their energy turns into heat when the waves are absorbed by the water. This heat source is described by a function f. To model the situation we consider a cross section of the water hose and consider the stationary heat equation. Let the hose's cross section be described by the domain $\Omega \in \mathbb{R}^2$ and let $u : \Omega \to \mathbb{R}$ be the temperature distribution. Then the temperature distribution u satisfies the boundary value problem

$$\begin{cases} -\nabla \cdot (a\nabla u) = f & \text{on } \Omega \subset \mathbb{R}^2, \\ a\vec{n} \cdot \nabla u = c(u_0 - u) & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Here a is the thermal conductivity coefficient of water, $\vec{n}$ is the outward pointing unit normal vector to the boundary $\partial\Omega$ of the domain Ω, c is the heat conductivity coefficient of the hose's walls, and u_0 is the temperature outside the hose. Typical values are $a_{\text{water}} = 0.6\frac{\text{W}}{\text{mK}}$ and $c = 100\frac{\text{W}}{\text{mK}}$.

Basic exercises (0 bonus points):

1. Find the weak formulation of the boundary value problem.

2. Assume that Ω is a circle of 1dm and write a Matlab program that solves the boundary value problem using the Finite Element Method. Take for now f as an (arbitrary) constant.

3. To obtain an a bit more interesting simulation we set the outside temperature to 250K. Ice has a different thermal conductivity coefficient than water, namely $a_{\text{ice}} = 2.2\frac{\text{W}}{\text{mK}}$. So the constant a will be depending on the temperature u now:

$$a(x) = \begin{cases} 2.2 & \text{if } u(x) \leq 273 \\ 0.6 & \text{if } u(x) > 273 \end{cases} . \tag{2.2}$$

   This gives a non-linear partial differential equation that we can solve e.g. with fixed point iteration. The fixed point iteration is the following procedure: Assume first $a = 0.6$ everywhere, then solve the boundary value problem. Change afterwards a to 2.2 at all node points where $u < 273$. Solve the boundary value problem again and adjust a according to the values of u. Repeat this procedure as long as a has to be adjusted, i.e. as long as there are triangles where a still has the "wrong" value.

   What constant value of the heat source function f describing the radiation is needed such that in the equilibrium state we have 50% ice and 50% water?

Extra exercises (1 bonus point each)

1. Implement source functions f that are not constant on Ω.

2. Describe a realistic physical model for absorption and scattering of microwaves in water that motivate a certain source function f. Give the function f and solve basic exercise 2 with this function.

3. Assume now again that the exterior temperature is 250K and that we have microwave radiation coming from above with some intensity I. Use now your physical model to describe how the water is heated by the microwaves. What intensity has to be chosen such that again in the equilibrium state we have 50% water and 50% ice in the hose?

2.2 Eigenvalue problem

Let A be an elliptic differential operator acting on functions defined on some domain $\Omega \in \mathbb{R}^2$. Then the eigenfunctions and eigenvalues of this operator give much insight to the properties of the operator in question. Moreover the solutions of the corresponding parabolic ($\dot{u} + Au = f$) and hyperbolic ($\ddot{u} + Au = f$) problems can be constructed from it.

It is well known that eigenvalues play an important role in classical and modern Physics. In this project we explore how the eigenvalue of a relatively general elliptic differential operator can be calculated numerically and how they change when the operator itself or the domain Ω is altered. The results shall be compared to results from Fourier Analysis.

Define the operator A by

$$Au := -\nabla \cdot (\alpha(x)\nabla u) + \vec{\beta}(x) \cdot \nabla u + \gamma(x)u, \quad u \in C^2(\Omega). \tag{2.3}$$

Consider the parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} + Au = f & \text{on } \Omega \times [0,T] \\ u = 0 & \text{on } \partial\Omega \times [0,T] \\ u = u_0 & \text{on } \Omega \times \{0\} \end{cases} \tag{2.4}$$

To find a solution to this problem, one can consider the corresponding eigenvalue problem

$$\begin{cases} Ae_n = \lambda_n e_n & \text{on } \Omega \\ e_n = 0 & \text{on } \partial\Omega \end{cases}, \quad n = 1, \ldots, \infty. \tag{2.5}$$

The spectral theorem implies that (2.5) has countably many solutions and that the eigenfunctions $\{e_n\}$ form an L_2-basis of $H^2(\Omega)$. Then a solution to the parabolic problem can be obtained by the ansatz

$$u(t,x,y) = \sum_{n=1}^{\infty} \varphi_n(t) e_n(x,y). \tag{2.6}$$

As an example one can consider the Laplace operator $\Delta := \nabla \cdot \nabla$ on a rectangular domain $\Omega = [0,L] \times [0,M]$. The eigenvalue problem with boundary condition $u = 0$ on $\partial\Omega$, $\Delta u_{ij} = \lambda_{ij} u_{ij}$ is solved by

$$\lambda_{ij} = \pi \left(\frac{i^2}{L^2} + \frac{j^2}{M^2} \right), \quad u_{ij}(x,y) = \sin\left(\frac{i\pi x}{M}\right) \sin\left(\frac{j\pi y}{L}\right), \quad i,j = 1, \ldots, \infty. \tag{2.7}$$

These solutions of the eigenvalue problem give then rise to solutions of the heat equation, $\dot{u} = \Delta u$, or the wave equation $\ddot{u} = \Delta u$.

Basic exercises (0 bonus points):

1. Write down the weak form of the eigenvalue problem (2.5) for an elliptic operator of the general form (2.3).

2. Write a Matlab program that solves this eigenvalue problem using the Finite Element Method. Assume $a = 1$, $\vec{\beta} = 0$, and $\gamma(x) \geq 0$ is an arbitrary function, e.g. $\gamma(x) = \frac{x}{5}$.

3. Use this Matlab program to calculate the eigenvalues of the Laplace operator on the domains

$$\begin{aligned}\Omega_1 &= [-1, 1] \times [-1, 1],\\ \Omega_2 &= [-3, 3] \times [-3, 3],\\ \Omega_3 &= [-2, -1] \times [-2, 2] \cup [-2, 2] \times [1, 2],\\ \Omega_4 &= \text{arbitrary set fulfilling}\, \Omega_1 \subsetneq \Omega_4 \subsetneq \Omega_2,\end{aligned}$$

   with the boundary conditions $u = 0$ on $\partial\Omega$. Compare the eigenvalues of the different domains.

   - Can you observe any laws how the eigenvalues change if one sclaes up a domain (cf. Ω_1 and Ω_2)?
   - Can you observe any laws how the eigenvalues change if one merges two domains (cf. Ω_3)?

4. Use a function $\gamma(x)$ instead of $\gamma \equiv 0$ and compare the eigenvalues on Ω_1.

2.2.1 Extra exercises (1 bonus point each):

1. Explain why all solutions of the eigenvalue problem (2.5) give a solution of the parabolic problem (2.4) and write down a formula for the solution $u(t, x, y)$ of (2.4) in terms of the initial data u_0, the eigenvalues λ_n of A, and the eigenfunctions e_n, $n \in \mathbb{N}$. (*Hint: Insert the ansatz (2.6) into the equation (2.4) to obtian an ODE for the coefficient functions φ_n. Use then the method of variation of constants to solve the inhomogeneous ODE.*)

2. Solve the eigenvalue problem for A with $\alpha = 1$, $\gamma = 0$, but $\vec{\beta} \neq 0$ on a circular domain Ω with the boundary conditions $u = 0$ on $\partial\Omega$. Calculate the eigenvalues for $\vec{\beta}_1 = \varepsilon(x, -y)$ and $\vec{\beta}_2 = \varepsilon(x, y)$ for some $\varepsilon \in \mathbb{R}\backslash\{0\}$. Note that β_1 is divergence free whereas β_2 is not. Compare the eigenvalues and consider the determinant and eigenvalues of the stiffness matrix.

3. Solve the parabolic problem (2.4) with your Matlab code on Ω_1. Take e.g. $\alpha \equiv 1$, $\beta \equiv 0$, $\gamma \equiv 0$, $f(t, x, y) = 10e^{-100(x^2+y^2)}$, and $u_0 \equiv 0$. Of course you can implement different parameters, too. (*Hint: Make sure that you normalize the obtained e_n such that $\|e_n\|_2 = 1$.*)

4. Discuss how the numerically calculated eigenvalues of A converge as the mesh parameter $h \to 0$.

2.3 Waves in a bath tub

Let $\Omega \in \mathbb{R}^2$ describe the area of a bath tub. The surface of the water (its amplitude $u(t,x,y)$ to be precise) in the tub can be modeled by the following boundary value problem:

$$\begin{cases} \ddot{u} - \Delta u = f & \text{on } \Omega \times [0,T], \\ \vec{n} \cdot \nabla u = 0 & \text{on } \partial\Omega \times [0,T], \\ u = u_0, & \text{on } \Omega \times \{0\}, \\ \dot{u} = v_0 & \text{on } \Omega \times \{0\} \end{cases} . \tag{2.8}$$

Here $\vec{n}$ is the outward pointing unit normal to the boundary $\partial\Omega$ of the domain Ω. A dot denotes the derivative with respect to t This problem will be considered as evolution problem.

The time evolution will be treated by a linearization whereas at each time step $u(t,\vec{x})$ will be calculated using the Finite Element Method.

2.3.1 Basic exercises (0 bonus points):

1. Write down the weak formulation of the boundary value problem (2.8). (Integrate over Ω, not over time.)

2. Discretize the weak form. First subdivide Ω into triangular cells and express the numerical approximation u_k of u as linear combination of hat functions φ_j with time dependent coefficients,

$$u_k(t,x,y) = \sum_{j=1}^{N} \xi_j(t)\varphi_j(x,y).$$

   Linearize then $\ddot{\xi}_j(t)$ by

$$\ddot{\xi}_j(t) \approx \frac{\xi_j(t+\Delta t) - 2\xi_j(t) + \xi(t-\Delta t)}{\Delta t^2}.$$

   Choose $\Delta t < h$ where h is the mesh parameter.

3. Implement a domain Ω in Matlab that resembles a bath tub and write a Matlab program that solves the wave equation on Ω. Take $f = 0$, v_0, and for u_0 the waves that a large drop would make when falling into the bathtub, for example

$$u_0(\vec{x}) = \frac{\cos(5\pi|\vec{x}|)}{1 + 10|\vec{x}|^2}. \tag{2.9}$$

It is important that the initial data u_0 and v_0 is correctly taken into account for the time discretization. This is done by first calculating the initial acceleration a_0. Let M be the mass matrix and S the stiffness matrix. Then $Ma_0 = f_0 - Su_0$. Then we can replace

$$u_{-1}(\vec{x}) := u(-\Delta t, \vec{x}) = u_0(\vec{x}) - \Delta t v_0(\vec{x}) + \frac{\Delta t^2}{2} a_0(\vec{x}).$$

2.3.2 Extra exercises (1 bonus point each):

1. Implement source functions f that depend on x, y, and t. What could they describe in the real world? Solve the boundary problem in this situation. (You can now take $u_0 \equiv 0$).

2. Take a non-trivial initial velocity distribution v_0 and solve the boundary value problem. What real-world situations could be described by the v_0 you consider?

3. Solve the boundary value problem for Dirichlet boundary conditions $u(t, \vec{x}) = 0$ for all $\vec{x} \in \partial\Omega$ and discuss the differences to the Neumann case. Take as source function

$$f(t, x, y) = \begin{cases} 1, & \text{if } t \in [0.5, 1] \text{ and } |(x, y)| \leq 0.2 \\ 0, & \text{else} \end{cases} \tag{2.10}$$

simulating a large droplet, and a continuous water jet

$$f(t, x, y) = \begin{cases} 1, & \text{if } |(x, y)| \leq 0.2 \\ 0, & \text{else} \end{cases} \quad . \tag{2.11}$$

(Adjust the dimensions of f, if necessary.)

3 Practical hints

The following hints might be helpful.

- In order to calculate the mass matrix, the stiffness matrix and the load vector integrals of φ_j, $\nabla\varphi_j$, and f have to be calculate. In S. Larsson and V. Thomé, *Partial Differential Equations with Numerical Methods*, Section 5.6 one finds useful formulas for this.

- On the course homepage the Matlab skeleton `MyPoissonSolver` is provided. It gives an idea of how an implementation of the Finite Element Method could be structured in Matlab.

- In particular `MyPoissonSolver` deals with general, mixed boundary conditions of the form
$$\vec{n} \cdot \nabla u(\vec{x}) = k(g(\vec{x}) - u(\vec{x})), \tag{3.1}$$
where k is a constant and g a function defined on the boundary. Neumann boundary conditions are obtained for $g = -u$ and Dirichlet boundary conditions for $k \to \infty$, thus by choosing a very large value for k.

- The domain and its subdivision into triangles can be nicely done using the PDE-App in Matlab. You start it by clicking right on top on the flag "Apps" and then select "PDE".