Department of
Mathematics
Göteborg

EXAMINATION FOR
LARGE AND SPARSE MATRIX PROBLEMS, TMA891/MMA610
2012-03-09

DATE: Friday March 9   TIME: 13.30 - 17.30   PLACE: MVH12

Examiner:	Ivar Gustafsson, tel: 772 10 94
Teacher on duty:	Ivar Gustafsson
Solutions:	Will be announced at the end of the exam on the board nearby room MVF21
Result:	Will be sent to you by April 1 at the latest Your marked examination can be received at the student's office at Mathematics Department, daily 12.30-13
Grades:	To pass requires 13 point, including bonus points from homework assignments Grades are evaluated by a formula involving also the computer exercises
Aids:	None (except dictionaries)

Instructions:
- State your methodology carefully. Motivate your statements clearly.
- Only write on one page of the sheet. Do not use a red pen. Do not answer more than one question per page.
- Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover.

GOOD LUCK!

Question 1.
a) Prove that a basic iterative method converges for any starting approximation if the spectral radius $\rho(R) < 1$, where R is the iteration matrix. You may use the fact that: $\forall\ \epsilon > 0\ \exists$ a $\|\cdot\|_*$-norm such that $\|R\|_* \leq \rho(R) + \epsilon$. **(2p)**
b) Present the red-black ordering of the 2D Poisson's problem. Which important property does the matrix have in this case? **(1p)**
c) If A is consistently ordered and $\omega \neq 0$ then the eigenvalus μ of R_J (the iteration matrix for Jacobi's method) and λ of $R_{SOR(\omega)}$ (the iteration matrix for the SOR method) are related by: $(\lambda + \omega - 1)^2 = \lambda\omega^2\mu^2$. Use this formula to prove that the Gauss-Seidel method is twice as fast as Jacobi's method when they converge. **(1p)**

Question 2.

a) If A is symmetric positive definite, the CG method minimizes $\|x_k - x\|_A$ over $x_k \in \mathcal{K}_k(A, b)$ (the Krylov space). Prove that this is equivalent to minimizing $\|r_k\|_{A^{-1}}$, where $r_k = b - Ax_k$ (the residual). **(2p)**

b) Present **two** important requirements on a good preconditioning matrix. Give an example of a good preconditioner **(2p)**

c) What is the order of the number of flops as a function of the stepzise h in the 2D Poisson's discetized matrix (the 5-point matrix) when the corresponding system of equations is solved by PCG and MIC preconditioning? **(1p)**

Question 3.

a) Present the multigrid V-cycle algorithm by a recursive function in a kind of programming language. **(3p)**

b) Explain the idea of damping in the solution operator in multigrid. Give an example of a good damping technique. **(2p)**

Question 4.

Consider the matrix $A = \begin{bmatrix} 4 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 4 & -1 & 0 & 0 & -1 & -1 \\ -1 & -1 & 4 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 4 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 4 & -1 \\ 0 & -1 & 0 & 0 & 0 & -1 & 4 \end{bmatrix}$.

a) Determine the amount of fill-in in the Cholesky factor corresponding to A with the given ordering **(1p)**

c) Determine a Reverse Curhill-McKee ordering for A and the fill-in in the Cholesky factor for this ordering. **(2p)**

d) Consider the minimum degree ordering for A. Choosing a good starting node (among a number of possible) is sometimes important. Consider two different starting nodes; one with largest eccentricity and one with smaller eccentricity. Determine a minimum degree ordering in both cases and the corresponding fill-ins in the Cholesky factors. **(3p)**

Question 5.

a) Present the basic steps in deriving the Lanczos-Rayleigh-Ritz method for solving large, sparse, symmetric eigenproblems. In particular explain why you get a tridiagonal matrix T such that $AQ = QT$ with Q orthogonal. Also explain how you get the approximate eigenvalues and eigenvectors **(3p)**

b) Explain why selective reorthogonalization is used and briefly indicate how it is performed. **(2p)**