Department of
Mathematics
Göteborg

EXAMINATION FOR
LARGE AND SPARSE MATRIX PROBLEMS, TMA891/MMA610
2011-03-17

DATE: Thursday March 17 TIME: 9.00 - 13.00 PLACE: MVF32

Examiner:	Ivar Gustafsson, tel: 772 10 94
Teacher on duty:	Ivar Gustafsson
Solutions:	Will be announced at the end of the exam on the board nearby room MVF21
Result:	Will be sent to you by April 1 at the latest Your marked examination can be received at the student's office at Mathematics Department, daily 12.30-13
Grades:	To pass requires 13 point, including bonus points from homework assignments Grades are evaluated by a formula involving also the computer exercises
Aids:	None (except dictionaries)

Instructions:
- State your methodology carefully. Motivate your statements clearly.
- Only write on one page of the sheet. Do not use a red pen. Do not answer more than one question per page.
- Sort your solutions by the order of the questions. Mark on the cover the questions you have answered. Count the number of sheets you hand in and fill in the number on the cover.

GOOD LUCK!

Question 1.
a) Define a regular splitting of a matrix and present a basic iterative method based on the splitting. Give **two** examples of useful splittings. **(2p)**
b) Prove that a basic iterative method **does not** converge for all starting approximations if the spectral radius of the iteration matrix is larger than or equal to 1. **(2p)**

Question 2.
a) Define the Krylov space used for solving large sparse systems of equations **Ax=b** in for instance the Lanczos method. How do you compute a useful basis for the Krylov space in the symmetric case? **(3p)**
b) Present **two** different strategies for deriving incomplete factorizations. What is the fundamental gain of using **modified** incomplete factorizations as preconditioners for the conjugate gradient method? **(2p)**

Question 3.
a) Describe the multigrid method by drawing a picture of a V-cycle and by presenting the three different operators involved. **(2p)**
b) Draw a picture of a full multigrid iteration. **(1p)**
c) For the 1D Poisson's problem we have the spectral decomposition $T = Z\Lambda Z^T$ of the corresponding tridiagonal matrix. Consider the weighted Jacobi's method with iteration matrix $R_\omega = I - \frac{\omega}{2}T$. For the error after m iterations we have $e^{(m)} = Z(I - \frac{\omega}{2}\Lambda)^m Z^T e^{(0)}$. The eigenvalues of T are $\lambda_j = 2(1-\cos\frac{\pi j}{N+1})$, $j = 1,\ 2,\ ...,\ N$. Explain by drawing a graph of the eigenvalues of R_ω that the choice $\omega = \frac{2}{3}$ is good in order for an efficient damping of the high frequency error components, compare with the choice $\omega = 1$. **(2p)**

Question 4.

Consider the matrix $A = \begin{bmatrix} 4 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 4 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 4 \end{bmatrix}$.

a) Indicate the envelope $Env(A)$ **(1p)**
b) Indicate the fill-in in the Cholesky factor corresponding to A with the given ordering **(1p)**
c) Determine a Reverse Curhill-McKee ordering for A and the fill-in in the Cholesky factor for this ordering. **(2p)**
d) Determine a minimum degree ordering for A and the fill-in in the Cholesky factor for this ordering. **(2p)**

Question 5. The Lanczos-Rayleigh-Ritz method ultimately computes an orthonormal basis Q in which the symmetric matrix A is represented by a tridiagonal matrix T, that is $AQ = QT$.
a) How do you get approximate eigenvalues and eigenvectors of A? **(1p)**
b) Use Weyl's theorem (see below) to prove a useful upper bound for the global error in the approximate eigenvalues. The upper bound should just consist of a single element from the matrix T. **(2p)**
Weyl's theorem: Let A and $\hat{A} = A + E$ be symmetric matrices with ordered eigenvalues $\{\alpha_i\}_{i=1}^n$ and $\{\hat{\alpha}_i\}_{i=1}^n$, respectively. Then $|\alpha_i - \hat{\alpha}_i| \leq \|E\|_2,\ i = 1,\ ...,\ n$.
c) Present **two** cases of misconvergence of the Lanczos-Rayleigh-Ritz method in exact arithmetics. **(1p)**
d) In the selective orthogonalization technique, against which vectors do we orthogonalize? **(1p)**