Applied optimization

Lecture 6

Linear integer optimization

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4 februari 2003

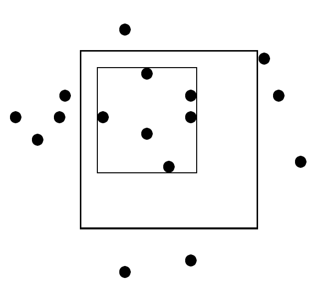
Screening of smear tests (granska cellprover)

- Prevent cancer in the womb (livmoderhalscancer)
- Regular examinations of all women above the age of 18
- Manual screening of each smear test using a microscope
- Prescreening using graphics processing $\Rightarrow \leq 50000$ points that must be manually screened
- ≈ 300 pictures/smear test (as few as possible $\Rightarrow$ more time for each picture)
- Optimization?
- Screen the pictures in the right order (automatically by the microscope)—not in this lecture

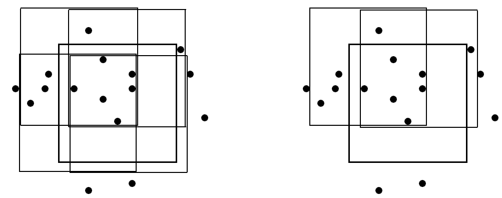
A smear test and an initial grid

- Totally 36 246 points and 392 squares (pictures)
- Can we decrease the *number* of pictures that have to be screened?

The smallest rectangle that covers all points in a square



Generation of alternative squares



The smear test and all square-candidates

- Totally 1610 square candidates
- Find the least number of squares to cover all the points

Mathematical model

The coefficient $\alpha_{kj} = \begin{cases} 1 & \text{if square } j \text{ covers point } k \\ 0 & \text{otherwise} \end{cases}$

The variable $x_j = \begin{cases} 1 & \text{if square } j \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}$

Cover each point with at least one square: (SET COVERING)

$$\begin{aligned} \min \quad & \sum_j x_j \\ \text{s.t.} \quad & \sum_j \alpha_{kj} x_j \geq 1 \quad \text{for all } k \\ & x_j \in \{0,1\} \quad \text{for all } j \end{aligned}$$

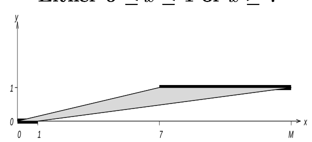
Smear test with “minimum” number of squares

- 36 246 points are covered by 339 squares
- $\approx 13\%$ fewer than the original 392

When are integer models needed?

- Products or raw materials are indivisible
- Logical constraints: "if A then B"; "A or B"
- Fixed costs
- Combinatorics (sequencing, allocation)
- On/off-decision to buy, invest, hire, generate electricity, ...

Either $0 \le x \le 1$ or $x \ge 7$

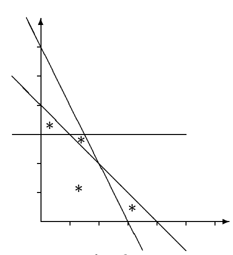


Let $M \gg 1: \quad x \le 1 + My,\ x \ge 7y,\ y \in \{0,1\}$

Variable x may only take the values 2, 45, 78 & 107

$$x = 2y_1 + 45y_2 + 78y_3 + 107y_4$$
$$y_1 + y_2 + y_3 + y_4 = 1$$
$$y_1, y_2, y_3, y_4 \in \{0,1\}$$

At least 2 of 3 constraints must be fulfilled



$$\begin{aligned} x_1 + x_2 &\le 4 && (1)\\ 2x_1 + x_2 &\le 6 && (2)\\ x_2 &\le 3 && (3)\\ \text{and } x_1, x_2 &\ge 0 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 &\le 4 + M(1-y_1) && (1)\\ 2x_1 + x_2 &\le 6 + M(1-y_2) && (2)\\ x_2 &\le 3 + M(1-y_3) && (3)\\ y_1 + y_2 + y_3 &\ge 2\\ y_1, y_2, y_3 &\in \{0,1\}\\ \text{and } x_1, x_2 &\ge 0 \end{aligned}$$

$*$ = feasible regions

$M \ge 2$

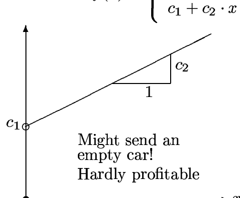
Fixed costs

x = the amount of a certain product to be sent

If $x > 0$ then the initial cost c_1 (e.g. car hire) is generated

Variable cost c_2 per unit sent

Total cost: $f(x) = \begin{cases} 0 & \text{if } x = 0\\ c_1 + c_2 \cdot x & \text{if } x > 0 \end{cases}$



$$\begin{aligned} \text{Let } M &= \text{car capacity}\\ y &= \begin{cases} 1 & \text{if } x > 0\\ 0 & \text{if } x = 0 \end{cases}\\ f(x,y) &= c_1 \cdot y + c_2 \cdot x\\ x &\le M \cdot y\\ x &\ge 0, \quad y \in \{0,1\} \end{aligned}$$

Other applications of integer optimization

- Facility location (new hospitals, shopping centres, etc.)
- Scheduling (on machines, personell, projects, for schools)
- Logistics (material- and warehouse control)
- Distribution (transportation of goods, buses for disabled persons)
- Production planning
- Telecommunication (network design, frequency allocation)
- VLSI-design

The combinatorial explosion

Assign n persons to carry out n jobs. # feasible solutions: $n!$

Assume that a feasible solution is evaluated in 10^{-9} seconds

n	2	5	8	10	100
$n!$	2	120	$4.0 \cdot 10^4$	$3.6 \cdot 10^6$	$9.3 \cdot 10^{157}$
⌈time⌉	10^{-8} s	10^{-6} s	10^{-4} s	10^{-2} s	10^{142} yrs

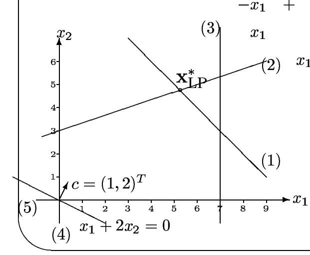
Complete enumeration of all solutions is **not** an efficient algorithm!

An algorithm exists that solves this problem in time $\mathcal{O}(n^4) \propto n^4$

n	2	5	8	10	100	1000
n^4	16	625	$4.1 \cdot 10^3$	10^4	10^8	10^{12}
⌈time⌉	10^{-7} s	10^{-6} s	10^{-5} s	$10^{-5} s$	10^{-1} s	17 min

Linear continuous optimization model

$$\begin{aligned} \max\ z_{\text{LP}} = \ & x_1 + 2x_2\\ \text{s.t.}\ & x_1 + x_2 \le 10 && (1)\\ & -x_1 + 3x_2 \le 9 && (2)\\ & x_1 \le 7 && (3)\\ & x_1, x_2 \ge 0 && (4,5) \end{aligned}$$



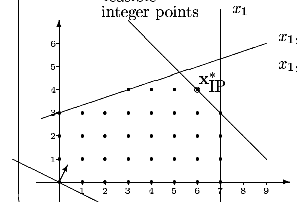
$$\mathbf{x}^*_{\text{LP}} = \begin{pmatrix} 21/4\\ 19/4 \end{pmatrix}$$

$$z^*_{\text{LP}} = 14 + \tfrac{3}{4}$$

Linear integer optimization model

$$\begin{aligned} \max\ z_{\text{IP}} = \ & x_1 + 2x_2\\ \text{s.t.}\ & x_1 + x_2 \le 10 && (1)\\ & -x_1 + 3x_2 \le 9 && (2)\\ & x_1 \le 7 && (3)\\ & x_1, x_2 \ge 0 && (4,5)\\ & x_1, x_2 \ \text{integer} \end{aligned}$$

• = feasible integer points

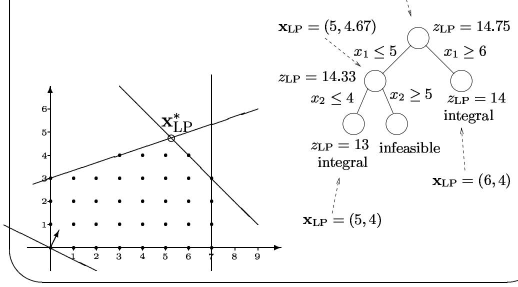


$$\mathbf{x}^*_{\text{IP}} = \begin{pmatrix} 6\\ 4 \end{pmatrix}$$

$$z^*_{\text{IP}} = 14 < z^*_{\text{LP}}$$

The branch–and–bound-algorithm

Relax integrality constraints ⇒ linear program ⇒ $\mathbf{x}_{\mathrm{LP}} = (5.25, 4.75)$



Algorithm specification

- Relaxation: Simplify [IP] by removing constraints
  - Get simple subproblems and optimistic estimates of z^*
- Branching strategy: rules for partitioning of X
  - Exclude non-integral solutions to relaxations
- Tree search strategy: defines in which order the B&B–tree is created and searched
  - Quickly find good feasible solutions; restrict the tree size
- Criteria for cutting off nodes: decide whether a subset of X should be partitioned or not
  - Avoid searching branches that cannot contain an optimal (a better) solution