Chalmers/GU
Mathematics

EXAM SOLUTION

TMA947/MAN280
APPLIED OPTIMIZATION

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Question 1

(the Simplex method)

a) By introducing a surplus-variable s_1 and a slack-variable s_2 the standard form of the problem is to

$$\begin{array}{llrrrrrl} \text{minimize} & z = & x_1 & +2x_2 & +3x_3 & & & \\ \text{subject to} & & 2x_1 & -5x_2 & +x_3 & -s_1 & & = 2, \\ & & 2x_1 & -x_2 & +2x_3 & & +s_2 & = 4, \\ & & x_1, & x_2, & x_3, & s_1, & s_2 & \geq 0. \end{array}$$

By introducing an artificial variable in the first constraint and solving the Phase I problem we get the BFS $\boldsymbol{x}_B = (x_1, s_2)$. This BFS also gives the optimal solution $\boldsymbol{x} = (x_1, x_2, x_3)^{\mathrm{T}} = (1, 0, 0)^{\mathrm{T}}$ to the original problem.

b) This is the unique optimal solution since the reduced costs of the non-basic variables $\boldsymbol{x}_N = (x_2, x_3, s_1)^{\mathrm{T}}$ are all strictly positive [$\tilde{\boldsymbol{c}}_N = (4.5, 2.5, 0.5)^{\mathrm{T}}$].

Question 2

(modelling)

Introduce the following variables:

$$x_{ij} = \text{the number of D from process } i = 1, 2 \text{ sent to demand center } j = 1, 2, 3,$$

$$z_i = \begin{cases} 1, & \text{if process } i = 1, 2 \text{ is used}, \\ 0, & \text{otherwise}. \end{cases}$$

The linear integer programming problem is then to

$$\begin{array}{ll} \text{minimize} & (c_A + 2c_B + 3c_C) \sum_{j=1}^{3} x_{1j} + (3c_A + 2c_B + c_C) \sum_{j=1}^{3} x_{2j} + \sum_{i=1}^{2} f_i z_i \\ \text{subject to} & \sum_{i=1}^{2} x_{ij} \geq d_j, \quad j = 1, 2, 3, \\ & \sum_{j=1}^{3} x_{ij} \leq p_i z_i, \quad i = 1, 2, \\ & z_i \in \mathbb{B}, \quad x_{ij} \in \mathbb{Z}_+, \quad i = 1, 2, \quad j = 1, 2, 3. \end{array}$$

Question 3

(Newton's algorithm)

a) Newton's equation:

$$x_{k+1} = x_k - \frac{\alpha x^{\alpha-1} - \exp(x)}{\alpha(\alpha-1)x^{\alpha-2} - \exp(x)}$$

b) The objective function of the problem is not convex in general [may be verified by analysing the sign of the Hessian $\alpha(\alpha-1)x^{\alpha-2} - \exp(x)$]. Since the convergence of the Newton method is local in nature, the method is most likely to converge to the nearest local minimum. The engineer thus wrongly assumes the global convergence of the Newton method.

c) Probably the simplest counter-example is obtained by taking $x_0 = 1$, $\alpha = 2$. These initial values cause the Newton's method to generate an oscillating sequence $x_{2k-1} = 0$, $x_{2k} = 1$, $k = 1, 2, \ldots$.

Question 4

(claims about optimality)

a) Additional property: the polyhedron is bounded.

Counter-example: the problem to maximize x_1 subject to $x_1 \geq 0$ has no optimal solution.

b) Additional property: f is weakly coercive.

Counter-example:

$$\text{minimize } f(x) = \begin{cases} -x, & x \leq 1, \\ 1/x, & x \geq 1 \end{cases}$$

is in C^1 on $\mathbb{R}$ and lower bounded. If we apply the steepest descent method on it, however, we obtain $\{x_k\} \to \infty$ while $\{f'(x_k)\} \to 0$.

c) Additional property: f and g are convex functions.

Counter-example: $f(x) = x^5 - 100x^3$; $g(x) = -x$; and $b = -6$ (that is, the constraint is $x \geq 6$).

The below plot shows the appearance of the function f in the interval $[-12, 12]$; clearly, the optimal solution to the constrained problem is $x^* \approx 7.5$, and $g(x^*) < b$ holds, but if we remove the constraint we see from the figure that there is no optimal solution to the problem—we may let $f(x)$ tend to minus infinity by letting x tend to minus infinity.

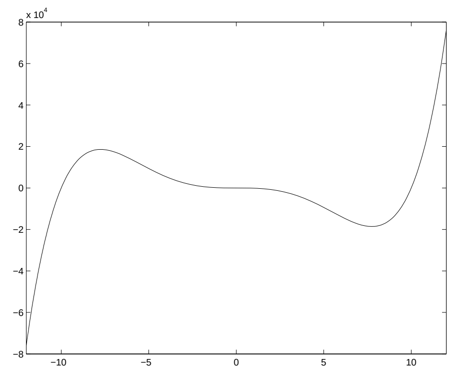

Figure 1: The function $f(x) = x^5 - 100x^3$ on an interval.

Question 5

(duality)

a) The only point that is feasible in the problem is the point $(x, y) = (0, 0)$ (easily verified graphically); thus it is the only locally and globally optimal solution. The problem is convex (both the objective function and the less-than-or-equal-to constraints are convex). It however does not satisfy the Slaters's CQ (there are no strictly feasible points), or LICQ (the gradients of the active constraints at the only feasible point are linearly dependent). An alternative argument is that the locally optimal solution $(0, 0)^{\mathrm{T}}$ is not a KKT point, which means that CQs cannot hold.

b) Introducing the Lagrange multipliers μ and λ for the constraints of the problem, we get

$$q(\lambda, \mu) = \inf_{(x,y)\in\mathbb{R}^2} \left\{ y + \lambda[(x-1)^2 + y^2 - 1] + \mu[(x+1)^2 + y^2 - 1] \right\}. \tag{1}$$

If $\lambda = \mu = 0$ we get $q(\lambda, \mu) = -\infty$; it remains thus to calculate q for assuming $\lambda + \mu > 0$. From the necessary (and sufficient in this convex case) optimality conditions we get:

$$\begin{cases} 2\lambda[x-1] + 2\mu[x-1] = 0 \\ 1 + 2\lambda y + 2\mu y = 0 \end{cases} \equiv \begin{cases} x = \dfrac{\lambda - \mu}{\lambda + \mu} \\ y = -\dfrac{1}{2(\lambda + \mu)}. \end{cases}$$

Substituting this into (1) we finally obtain

$$q(\lambda, \mu) = -\frac{1}{4(\lambda + \mu)} - \frac{(\lambda - \mu)^2}{\lambda + \mu}.$$

c) Show that the strong duality holds, that is, $z^* = \sup_{\lambda \in \mathbb{R}^2_+} q(\lambda)$, where z^* is the optimal value of the primal problem.

Clearly, in our case $z^* = 0$. Thus, by the weak duality, or from the explicit formula for the dual function, we have that for all $(\lambda, \mu) \in \mathbb{R}^2_+$ it holds that $q(\lambda, \mu) < 0$. Still, $\lim_{\lambda \to +\infty} q(\lambda, \lambda) = 0$, which means that $\sup_{(\lambda,\mu)\in\mathbb{R}^2_+} q(\lambda, \mu) = 0 = z^*$.

Question 6

(convexity)

a) (1) We utilize the following characterization of convexity of f on $\mathbb{R}^n$:

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x}), \qquad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n.$$

It follows that if $\boldsymbol{x}$ and $\boldsymbol{y}$ are such that $\nabla f(\boldsymbol{x})^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x}) \geq 0$ then $f(\boldsymbol{y}) \geq f(\boldsymbol{x})$ also holds; hence, f is pseudo-convex on $\mathbb{R}^n$.

(2) The function in the below figure is of a form often referred to as "unimodal," that is, it has a unique minimum and it increases both to the right and left of this minimum. If such a function is differentiable then it is also pseudo-convex (check this!).

It is however not convex.

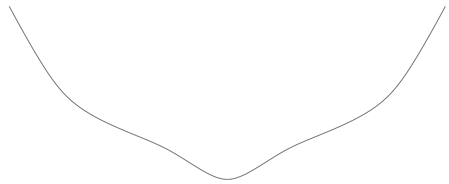

Figure 2: A unimodal function.

b) The result in the direction of $\Longrightarrow$ (the necessary condition) is true for every differentiable function. The result in the direction of $\Longleftarrow$ (the sufficient condition) follows immediately from the definition of pseudo-convexity.

c) (1) This result is Proposition 3.48 in the Course Notes.

(2) A unimodal function has convex level sets, and so the example in the above figure works as a conuter-example here as well.

Question 7

(Lagrangian duality)

The proof of the consistency of the global optimality conditions be found in Theorem 7.6 in the Course Notes.
