EXERCISE 7: THE GEOMETRY OF LINEAR PROGRAMMING

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EXERCISE 1 (standard form). Transform the linear program

$$\begin{aligned}
\text{minimize} \quad & z = x_1 - 5x_2 - 7x_3 \\
\text{subject to} \quad & 5x_1 - 2x_2 + 6x_3 \geq 5, && (1)\\
& 3x_1 + 4x_2 - 9x_3 = 3, && (2)\\
& 7x_1 + 3x_2 + 5x_3 \leq 9, && (3)\\
& x_1 \geq -2,
\end{aligned}$$

into standard form! □

EXERCISE 2 (standard form). Consider the linear program

$$\begin{aligned}
\text{minimize} \quad & z = 5x_1 + 3x_2 - 7x_3 \\
\text{subject to} \quad & 2x_1 + 4x_2 + 6x_3 = 11, \\
& 3x_1 - 5x_2 + 3x_3 + x_4 = 11, \\
& x_1, \quad x_2, \quad x_4 \geq 0.
\end{aligned}$$

(a) Show how to transform this problem into standard form by eliminating the unrestricted variable x_3.
(b) Why cannot this technique be used to eliminate variables with non-negativity restrictions?

□

EXERCISE 3 (basic feasible solutions). Suppose that a linear program includes a free variable x_j. When transforming this problem into standard form, x_j is replaced by

$$x_j = x_j^+ - x_j^-,$$
$$x_j^+, x_j^- \geq 0.$$

Show that no basic feasible solution can include both x_j^+ and x_j^- as non-zero basic variables. □

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EXERCISE 4 (equivalent systems). Consider the system of equations

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \qquad i = 1, \dots, m. \tag{1}$$

Show that this system is equivalent to the system

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, \quad i = 1, \dots, m, \tag{2.a}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_j \geq \sum_{i=1}^{m} b_i. \tag{2.b}$$

□

EXERCISE 5 (application of Farkas' Lemma). In a paper submitted for publication in an operations research journal, the author considered the set

$$P = \left\{ \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in \mathbb{R}^{n+m} \;\middle|\; \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} \geq \boldsymbol{c}; \quad \boldsymbol{x} \geq \boldsymbol{0}^n; \quad \boldsymbol{y} \geq \boldsymbol{0}^m \right\},$$

where $\boldsymbol{A}$ is an $m \times n$ matrix, $\boldsymbol{B}$ a positive semi-definite $m \times m$ matrix and $\boldsymbol{c} \in \mathbb{R}^m$. The author explicitly assumed that the set P is compact in $\mathbb{R}^{n+m}$. A reviewer of the paper pointed out that the only compact set of the above form is the empty set. Prove the reviewer's assertion! □