EXERCISE 8: THE SIMPLEX METHOD

NICLAS ANDRÉASSON

EXERCISE 1 (Checking feasibility: Phase I). Consider the system

$$\begin{array}{rrrl} 3x_1 & +2x_2 & -x_3 & \leq -3, \\ -x_1 & -x_2 & +2x_3 & \leq -1, \\ x_1, & x_2, & x_3 & \geq 0. \end{array}$$

Show that this system is infeasible! □

The Simplex algorithm

Step 0. (initialization): Assume that $\boldsymbol{x}^{\mathrm{T}} = (\boldsymbol{x}_B^{\mathrm{T}}, \boldsymbol{x}_N^{\mathrm{T}})$ is a BFS corresponding to the partition $\boldsymbol{A} = (\boldsymbol{B}, \boldsymbol{N})$.

Step 1. (entering variable, pricing): Calculate the reduced costs of the non-basic variables,

$$(\tilde{\boldsymbol{c}}_N)_j = (\boldsymbol{c}_N^{\mathrm{T}} - \boldsymbol{c}_B^{\mathrm{T}} \boldsymbol{B}^{-1} \boldsymbol{N})_j, \quad j = 1, \dots, n-m.$$

If $(\tilde{\boldsymbol{c}}_N)_j \geq 0$ for all $j = 1, \dots, n-m$ then stop; $\boldsymbol{x}$ is then optimal. Otherwise choose $(\boldsymbol{x}_N)_j$, where

$$j \in \arg \underset{j \in \{1, \dots, n-m\}}{\operatorname{minimum}} \{(\tilde{\boldsymbol{c}}_N)_j\},$$

to enter the basis.

Step 2. (leaving variable): If

$$\boldsymbol{B}^{-1} \boldsymbol{N}_j \leq \boldsymbol{0}^m,$$

then the problem is unbounded, stop; $((\boldsymbol{B}^{-1}\boldsymbol{N}_j)^{\mathrm{T}}, \boldsymbol{e}_j^{\mathrm{T}})^{\mathrm{T}}$ is then a direction of unboundness. Otherwise choose $(\boldsymbol{x}_B)_i$, where

$$i \in \arg \underset{i \in \{i \mid (\boldsymbol{B}^{-1}\boldsymbol{N}_j)_i > 0\}}{\operatorname{minimum}} \frac{(\boldsymbol{B}^{-1}\boldsymbol{b})_i}{(\boldsymbol{B}^{-1}\boldsymbol{N}_j)_i},$$

to leave the basis.

Step 3. (change basis): Construct a new partition by swapping $(\boldsymbol{x}_B)_i$ with $(\boldsymbol{x}_N)_j$. Go to Step 1.

Date: February 20, 2004.

EXERCISE 2 (The Simplex algorithm: Phase I & II). Consider the linear program

$$\begin{array}{lrrrl} \text{minimize} & z = 3x_1 & +2x_2 & +x_3 & \\ \text{subject to} & 2x_1 & & +x_3 & \geq 3, \\ & 2x_1 & +2x_2 & +x_3 & = 5, \\ & x_1, & x_2, & x_3 & \geq 0. \end{array}$$

(a) Solve the linear program by using the Simplex algorithm with Phase I & II!
(b) Is the solution obtained unique?

□

EXERCISE 3 (Sensitivity analysis: Perturbations in the objective function). Consider the linear program

$$\begin{array}{llrrrrl} \text{maximize} & z = & -x_1 & +18x_2 & +c_3x_3 & +c_4x_4 & \\ \text{subject to} & & x_1 & +2x_2 & +3x_3 & +4x_4 & \leq 3, \\ & & -3x_1 & +4x_2 & -5x_3 & -6x_4 & \leq 1, \\ & & x_1, & x_2, & x_3, & x_4 & \geq 0. \end{array}$$

Find the values of c_3 and c_4 such that the basic solution that corresponds to the partition $\boldsymbol{x}_B = (x_1, x_2)^{\mathrm{T}}$ is an optimal basic feasible solution to the problem! □

EXERCISE 4 (Sensitivity analysis: Perturbations in the right-hand side). Consider the linear program

$$\begin{array}{lrrrl} \text{minimize} & z = -x_1 & +2x_2 & +x_3 & \\ \text{subject to} & 2x_1 & +x_2 & -x_3 & \leq 7, \\ & -x_1 & +2x_2 & +3x_3 & \geq 3 + \delta, \\ & x_1, & x_2, & x_3 & \geq 0. \end{array}$$

(a) Show that the basic solution that corresponds to the partition $\boldsymbol{x}_B = (x_1, x_3)^{\mathrm{T}}$ is an optimal solution to the problem when $\delta = 0$!
(b) Find the values of the perturbation $\delta \in \mathbb{R}$ such that the above BFS is optimal!

□

EXERCISE 5 (equivalent problems). Consider the linear program in standard form,

$$\begin{array}{ll} \text{minimize} & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n. \end{array}$$

Suppose that at a given step of the Simplex algorithm, there is only one possible entering variable, $(\boldsymbol{x}_N)_j$. Also assume that the current BFS is non-degenerate. Show that $(\boldsymbol{x}_N)_j > 0$ in any optimal solution! □