EXERCISE 9-10: LINEAR PROGRAMMING DUALITY AND INTEGER MODELLING

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Rules for the construction of the LP dual problem

primal/dual constraint		**dual/primal variable**
canonical inequality	$\iff$	≥ 0
non-canonical inequality	$\iff$	≤ 0
equality	$\iff$	unrestricted

EXERCISE 1 (Constructing the LP dual). Consider the linear program

$$
\begin{array}{lrll}
\text{maximize} & z = 6x_1 - 3x_2 - 2x_3 + 5x_4 & & \\
\text{subject to} & 4x_1 + 3x_2 - 8x_3 + 7x_4 = 11, & & (1) \\
& 3x_1 + 2x_2 + 7x_3 + 6x_4 \geq 23, & & (2) \\
& 7x_1 + 4x_2 + 3x_3 + 2x_4 \leq 12, & & (3) \\
& x_1, \quad x_2 \geq 0, & & \\
& x_3 \leq 0, & & \\
& x_4 \quad \text{free}. & &
\end{array}
$$

Construct the linear programming dual! □

EXERCISE 2 (Constructing the LP dual). Consider the linear program

$$
\begin{array}{ll}
\text{minimize} & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\
\text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\
& \boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u}.
\end{array}
$$

(a) Construct the linear programming dual!
(b) Show that the dual problem is always feasible (independent of $\boldsymbol{A}$, $\boldsymbol{b}$, $\boldsymbol{l}$, and $\boldsymbol{u}$)!

□

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EXERCISE 3 (Constructing an optimal dual solution from an optimal BFS). Consider the linear program in standard form

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}^n. \end{aligned} \tag{P}$$

Assume that an optimal BFS, $\mathbf{x}^* = (\mathbf{x}_B^{\mathrm{T}}, \mathbf{x}_N^{\mathrm{T}})^{\mathrm{T}}$, is given by the partition $\mathbf{A} = (\mathbf{B}, \mathbf{N})$. Show that

$$\mathbf{y} = (\mathbf{B}^{-1})^{\mathrm{T}}\mathbf{c}_B$$

is an optimal solution to the LP dual problem! □

EXERCISE 4 (Application of the Weak and Strong Duality Theorems). Consider the linear program

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}^n, \end{aligned} \tag{P}$$

and the perturbed problem to

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \tilde{\mathbf{b}}, \\ & \mathbf{x} \geq \mathbf{0}^n. \end{aligned} \tag{P'}$$

Show that if (P) has an optimal solution, then the perturbed problem (P') cannot be unbounded (independent of $\tilde{\mathbf{b}}$)! □

EXERCISE 5 (Application of the Weak and Strong Duality Theorems). Consider the linear program

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}. \end{aligned} \tag{P}$$

Assume that the objective function vector $\mathbf{c}$ cannot be written as a linear combination of the rows of $\mathbf{A}$. Show that (P) cannot have an optimal solution! □

EXERCISE 6 (Application of the Weak and Strong Duality Theorems). Consider the linear program

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}^n. \end{aligned} \tag{P}$$

Construct a polyhedron that equals the set of optimal solutions to (P)! □

EXERCISE 7 (Application of the Weak and Strong Duality Theorems). Consider the linear program

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}^n. \end{aligned} \tag{P}$$

Let $\mathbf{x}^*$ be an optimal solution to (P) with the optimal objective function value z^*, and let $\mathbf{y}^*$ be an optimal solution to the LP dual of (P). Show that

$$z^* = (\mathbf{y}^*)^{\mathrm{T}}\mathbf{A}\mathbf{x}^*.$$

□

EXERCISE 8 (Linear programming primal-dual optimality conditions). Consider the linear program

$$\begin{array}{lrrrrrl} \text{maximize} & z = & -4x_2 & +3x_3 & +2x_4 & -8x_5 & \\ \text{subject to} & 3x_1 & +x_2 & +2x_3 & +x_4 & & = 3, \\ & x_1 & -x_2 & & +x_4 & -x_5 & \geq 2, \\ & x_1, & x_2, & x_3, & x_4, & x_5 & \geq 0. \end{array}$$

Find an optimal solution by using the LP primal-dual optimality conditions! □

EXERCISE 9 (Linear programming primal-dual optimality conditions). Consider the linear program (the continuous knapsack problem)

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{a}^{\mathrm{T}}\mathbf{x} \leq b, \\ & \mathbf{x} \leq \mathbf{1}^n, \\ & \mathbf{x} \geq \mathbf{0}^n, \end{aligned} \tag{P}$$

where $\mathbf{c} > \mathbf{0}^n$, $\mathbf{a} > \mathbf{0}^n$, $b > 0$, and

$$\frac{c_1}{a_1} \geq \frac{c_2}{a_2} \geq \cdots \geq \frac{c_n}{a_n}.$$

Show that the feasible solution $\mathbf{x}$ given by

$$x_j = 1,\ j = 1, \dots, r-1, \quad x_r = \frac{b - \sum_{j=1}^{r-1} a_j}{a_r}, \quad x_j = 0,\ j = r+1, \dots, n,$$

where r is such that $\sum_{j=1}^{r-1} a_j < b$ and $\sum_{j=1}^{r} a_j > b$, is an optimal solution! □

EXERCISE 10 (KKT versus LP primal-dual optimality conditions). Consider the linear program

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \end{aligned} \tag{P}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{c} \in \mathbb{R}^n$, and $\mathbf{b} \in \mathbb{R}^m$. Show that the KKT optimality conditions are equivalent to the LP primal-dual optimality conditions! □

EXERCISE 11 (Lagrangian primal-dual versus LP primal-dual). Consider the linear program

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}. \end{aligned}$$

Show that the Lagrangian primal-dual optimality conditions are equivalent to the LP primal-dual optimality conditions! □

EXERCISE 12 (Integer modelling). A company has decided to establish themselves in Göteborg. In order to know where to place their stores, they have made a market survey in which the population has been divided into m customer areas, with c_i potential customers in each area. The company has surveyed n possible store locations, and the maximum customer capacity of a store in location j is given by p_j. A store is said to belong to a customer area's *primary* region if the store and the area are really close, and customers will always prefer stores in the primary area. The set P_i lists the stores in the primary region for customer area i. If there is no store in the primary area, or if all these stores are full, then some customers may choose to walk a bit to buy an item, whereas others will simply not shop. The company assumes that 50% of the potential customers who cannot be served within the primary region will go to a store within the *secondary* region, while the other 50% will go home without shopping. The stores in the secondary area of region i are given by the set S_i. The annual cost of running a store in location j is r_j, and each customer served will give an annual income of q. Formulate an integer linear programming model for the problem! □