Lecture 10: Linear programming duality

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The canonical primal–dual pair

$\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, and $\boldsymbol{c} \in \mathbb{R}^n$

$$\begin{aligned} \text{maximize} \quad & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned} \tag{1}$$

and

$$\begin{aligned} \text{minimize} \quad & w = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \\ \text{subject to} \quad & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \geq \boldsymbol{c}, \\ & \boldsymbol{y} \geq \boldsymbol{0}^m \end{aligned} \tag{2}$$

The dual of the LP in standard form

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned} \tag{P}$$

and

$$\begin{aligned} \text{maximize} \quad & w = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \\ \text{subject to} \quad & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}, \\ & \boldsymbol{y} \ \text{ free} \end{aligned} \tag{D}$$

Rules for formulating dual LPs

- We say that an inequality is *canonical* if it is of $\leq$ [respectively, $\geq$] form in a maximization [respectively, minimization] problem.
- We say that a variable is *canonical* if it is ≥ 0.
- The rule is that the dual variable [constraint] for a primal constraint [variable] is canonical if the other one is canonical. If the direction of a primal constraint [sign of a primal variable] is the opposite from the canonical, then the dual variable [dual constraint] is also opposite from the canonical.

- Further, the dual variable [constraint] for a primal equality constraint [free variable] is free [an equality constraint].
- Summary:

primal/dual constraint		**dual/primal variable**
canonical inequality	$\iff$	≥ 0
non-canonical inequality	$\iff$	≤ 0
equality	$\iff$	unrestricted (free)

Weak Duality Theorem 11.4

- *If* $\boldsymbol{x}$ *is a feasible solution to* (P) *and* $\boldsymbol{y}$ *a feasible solution to* (D), *then* $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \geq \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$.
- Similar relation for the primal–dual pair (2)–(1): the max problem never has a higher objective value.
- *Proof.* $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \geq (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y})^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$. □
- Corollary: If $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$ for a feasible primal–dual pair $(\boldsymbol{x}, \boldsymbol{y})$ then they must be optimal.

Strong Duality Theorem 11.6

- In the compendium, strong duality is established for the pair (P) and (D). Here, we establish the result for the pair (1), (2).
- *If one of the problems* (1) *and* (2) *has a finite optimal solution, then so does its dual, and their optimal objective values are equal.*
- *Proof.* The idea behind the proof is as follows. We suppose first that it is the primal maximization problem that has a finite optimal solution. (This is without loss of generality.) We then state an optimal BFS representing an optimal extreme point $\boldsymbol{x}$, and

thereafter establish (a) that from the optimality condition that the reduced costs are non-positive, we can construct a dual vector of the form $\boldsymbol{y}^{\mathrm{T}} = \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{B}^{-1}$ which is feasible in (2). (b) We show that the objective values $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ and $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$ are equal. Hence, the dual vector must be optimal in its problem.

- (a) Suppose we have added slacks $\boldsymbol{s} \in \mathbb{R}^m$, and represented an optimal extreme point $\boldsymbol{x}$ through a basic/non-basic partitioning of $(\boldsymbol{x}, \boldsymbol{s})$ and correspondingly of $(\boldsymbol{A}, \boldsymbol{I}^m)$ and $(\boldsymbol{c}, \boldsymbol{0}^m)$. Suppose that the basis is optimal. Then, all the reduced costs of the $\boldsymbol{x}$ and $\boldsymbol{s}$ variables are non-positive.

- Hence, $\bar{\boldsymbol{c}}_{\boldsymbol{x}}^{\mathrm{T}} = \boldsymbol{c}^{\mathrm{T}} - \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{A} \leq (\mathbf{0}^n)^{\mathrm{T}}$ and $\bar{\boldsymbol{c}}_{\boldsymbol{s}} = (\mathbf{0}^m)^{\mathrm{T}} - \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{I}^m \leq (\mathbf{0}^m)^{\mathrm{T}}$.
- Now, let us define a dual vector as follows: $\boldsymbol{y}^{\mathrm{T}} := \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{B}^{-1}$.
- We notice that this choice is identical to that which we saw was provided in the pricing step of the Simplex method. Therefore, we can say that this vector is provided for free from having used the Simplex method in finding an optimal BFS.
- Then, $\boldsymbol{c}^{\mathrm{T}} - \boldsymbol{y}^{\mathrm{T}}\boldsymbol{A} \leq (\mathbf{0}^n)^{\mathrm{T}}$ and $-\boldsymbol{y}^{\mathrm{T}}\boldsymbol{I}^m \leq (\mathbf{0}^m)^{\mathrm{T}}$.
- In other words, $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \geq \boldsymbol{c}$ and $\boldsymbol{y} \geq \mathbf{0}^m$, that is, $\boldsymbol{y} := (\boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{B}^{-1})^{\mathrm{T}}$ is feasible in (2).

- (b) Note that at the BFS chosen, $z = \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = w$. By the construction of the dual vector $\boldsymbol{y}$ in this way, it will *always* have the same value as the primal BFS! In particular, it has here the same objective value as the primal optimal one. Use the above Corollary. □

Complementary Slackness Theorem 11.12

- *Let* $\boldsymbol{x}$ *be a feasible solution to* (1) *and* $\boldsymbol{y}$ *a feasible solution to* (2). *Then* $\boldsymbol{x}$ *is optimal to* (1) *and* $\boldsymbol{y}$ *optimal to* (2) *if and only if*

$$x_j(c_j - \boldsymbol{y}^{\mathrm{T}}\boldsymbol{A}_{\cdot j}) = 0, \qquad j = 1, \dots, n, \qquad (3a)$$

$$y_i(\boldsymbol{A}_{i\cdot}\boldsymbol{x} - b_i) = 0, \qquad i = 1 \dots, m, \qquad (3b)$$

where $\boldsymbol{A}_{\cdot j}$ *is the* j*th column of* $\boldsymbol{A}$ *and* $\boldsymbol{A}_{i\cdot}$ *the* i*th row of* $\boldsymbol{A}$.

- *Proof.* From Weak and Strong Duality we have both that

$$\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \geq (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y})^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$$

and that $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}$ holds. Since we have equality throughout above, we have that $0 = [\boldsymbol{c} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y}]^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{y}^{\mathrm{T}}[\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}]$. Since each term is sign restricted, each one must be zero. We are done with this direction.

- The converse argument follows similarly: weak duality plus complementarity implies strong duality. □

Necessary and sufficient optimality conditions: Strong duality, the global optimality conditions, and the KKT conditions are equivalent for LP

- We have seen above that the following statement characterizes the optimality of a primal–dual pair $(\boldsymbol{x}, \boldsymbol{y})$:
- $\boldsymbol{x}$ is feasible in (1), $\boldsymbol{y}$ is feasible in (2), and complementarity holds.
- In other words, we have the following result (think of the KKT conditions!):

- Theorem 11.14: *Take a vector* $\boldsymbol{x} \in \mathbb{R}^n$. *For* $\boldsymbol{x}$ *to be an optimal solution to the linear program* (1), *it is both necessary and sufficient that*

  (a) $\boldsymbol{x}$ *is a feasible solution to* (1);

  (b) *corresponding to* $\boldsymbol{x}$ *there is a dual feasible solution* $\boldsymbol{y} \in \mathbb{R}^m$ *to* (2); *and*

  (c) $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *together satisfy complementarity* (3). □
- This is precisely the same as the KKT conditions!
- Those who wishes to establish this—note that there are no multipliers for the "$\boldsymbol{x} \geq \boldsymbol{0}^n$" constraints, and in the KKT conditions there are. Introduce such a multiplier vector and see that it can later be eliminated.

- Further: suppose that $\boldsymbol{x}$ and $\boldsymbol{y}$ are feasible in (1) and (2). Then, the following are equivalent:

  (a) $\boldsymbol{x}$ and $\boldsymbol{y}$ have the same objective value;

  (b) $\boldsymbol{x}$ and $\boldsymbol{y}$ solve (1) and (2);

  (c) $\boldsymbol{x}$ and $\boldsymbol{y}$ satisfy complementarity.

The Simplex method and the global optimality conditions

- The Simplex method is remarkable in that it satisfies two of the three conditions at every BFS, and the remaining one is satisfied at optimality:
- $\boldsymbol{x}$ is feasible after Phase-I has been completed.
- $\boldsymbol{x}$ and $\boldsymbol{y}$ always satisfy complementarity. Why? If x_j is in the basis, then it has a zero reduced cost, implying that the dual constraint j has no slack. If the reduced cost of x_j is non-zero, then its value is zero.

- The feasibility of $\boldsymbol{y}^{\mathrm{T}} = \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{B}^{-1}$ is not fulfilled until we reach an optimal BFS. How is the incoming criterion related to this? We introduce as an incoming variable that variable which has the best reduced cost. Since the reduced cost measures the dual feasibility of $\boldsymbol{y}$, this means that we select the most violated dual constraint; at the new BFS, that constraint is then satisfied (since the reduced cost then is zero). The Simplex method hence works to try to satisfy dual feasibility!

Farkas' Lemma revisited

- *Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{b}$ an $m \times 1$ vector. Then exactly one of the systems*

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b}, \\ \boldsymbol{x} &\geq \boldsymbol{0}^n, \end{aligned} \tag{I}$$

and

$$\begin{aligned} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} &\leq \boldsymbol{0}^n, \\ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} &> 0, \end{aligned} \tag{II}$$

has a feasible solution, and the other system is inconsistent.

- *Proof.* If (I) has a solution $\boldsymbol{x}$, then

$$\boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} > 0.$$

But $\boldsymbol{x} \geq \boldsymbol{0}^n$, so $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{0}^n$ cannot hold, which means that (II) is infeasible.

- Assume that (II) is infeasible. Consider the linear program

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \\ \text{subject to} \quad & \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{0}^n, \\ & \boldsymbol{y} \quad \text{free}, \end{aligned} \tag{4}$$

and its dual program

$$\begin{aligned} \text{minimize} \quad & (\boldsymbol{0}^n)^{\mathrm{T}}\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n. \end{aligned} \tag{5}$$

Since (II) is infeasible, $\boldsymbol{y} = \boldsymbol{0}^m$ is an optimal solution to (4). Hence the Strong Duality Theorem 11.6 gives that there exists an optimal solution to (5). This solution is feasible to (I). □

Decentralized planning

- Consider the following profit maximization problem:

$$\text{maximize } z = \boldsymbol{p}^{\mathrm{T}}\boldsymbol{x} = \sum_{i=1}^{m} \boldsymbol{p}_i^{\mathrm{T}}\boldsymbol{x}_i,$$

$$\text{s.t.} \begin{pmatrix} \boldsymbol{B}_1 & & & \\ & \boldsymbol{B}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{B}_m \\ & & \boldsymbol{C} & \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_m \end{pmatrix} \leq \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{b}_m \\ \boldsymbol{c} \end{pmatrix},$$

$$\boldsymbol{x}_i \geq \boldsymbol{0}^{n_i}, \quad i = 1, \ldots, m,$$

for which we have the following interpretation:

- We have m independent subunits, responsible for finding their optimal production plan.
- While they are governed by their own objectives, we (the Managers) want to solve the overall problem of maximizing the company's profit.
- The constraints $\boldsymbol{B}_i\boldsymbol{x}_i \leq \boldsymbol{b}_i$, $\boldsymbol{x}_i \geq \boldsymbol{0}^{n_i}$ describe unit i's own production limits, when using their own resources.

- The units also use limited resources that are the same.
- The resource constraint is difficult as well as unwanted to enforce directly, because it would make it a *centralized planning* process.
- We want the units to maximize their own profits individually.
- But we must also make sure that they do not violate the resource constraints $\boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{c}$.
- (This constraint is typically of the form $\sum_{i=1}^{m} \boldsymbol{C}_i\boldsymbol{x}_i \leq \boldsymbol{c}$.)
- How?

- ANSWER: Solve the LP dual problem!
- Generate from the dual solution the dual vector $\boldsymbol{y}$ for the joint resource constraint.
- Introduce an *internal price* for the use of this resource, equal to this dual vector.
- Let each unit optimize their own production plan, with an additional cost term.
- This will then be a *decentralized planning* process.

- Each unit i will then solve their own LP problem to

$$\begin{aligned} \underset{\boldsymbol{x}_i}{\text{maximize}}\ & [\boldsymbol{p}_i - \boldsymbol{C}_i^{\mathrm{T}}\boldsymbol{y}]^{\mathrm{T}}\boldsymbol{x}_i, \\ \text{subject to}\ & \boldsymbol{B}_i\boldsymbol{x}_i \leq \boldsymbol{b}_i, \\ & \boldsymbol{x} \geq \boldsymbol{0}^{n_i}, \end{aligned}$$

  resulting in an optimal production plan!

- *Decentralized planning*, is related to *Dantzig–Wolfe* decomposition, which is a general technique for solving large-scale LP by solving a sequence of smaller LP:s.