Lecture 9: The Simplex method

Michael Patriksson

17 February 2004

An algebraic derivation of the pricing step

$$
\begin{aligned}
z^* = \quad & \text{infimum } \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} && = \quad \text{infimum } \boldsymbol{c}_{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{x}_{\boldsymbol{B}} + \boldsymbol{c}_{\boldsymbol{N}}^{\mathrm{T}}\boldsymbol{x}_{\boldsymbol{N}} \\
& \text{subject to } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, && \quad \text{subject to } \boldsymbol{B}\boldsymbol{x}_{\boldsymbol{B}} + \boldsymbol{N}\boldsymbol{x}_{\boldsymbol{N}} = \boldsymbol{b}, \\
& \qquad \boldsymbol{x} \geq \boldsymbol{0}^n && \qquad \boldsymbol{x}_{\boldsymbol{B}} \geq \boldsymbol{0}^m;\ \boldsymbol{x}_{\boldsymbol{N}} \geq \boldsymbol{0}^{n-m} \\
= \boldsymbol{c}_{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{b} + \quad & \text{infimum } [\boldsymbol{c}_{\boldsymbol{N}}^{\mathrm{T}} - \boldsymbol{c}_{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{N}]\boldsymbol{x}_{\boldsymbol{N}} \\
& \text{subject to } \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_{\boldsymbol{N}} \geq \boldsymbol{0}^m, \\
& \qquad \boldsymbol{x}_{\boldsymbol{N}} \geq \boldsymbol{0}^{n-m}
\end{aligned}
$$

- $\boldsymbol{x}_{\boldsymbol{N}} = \boldsymbol{0}^{n-m}$ feasible. Let $\bar{\boldsymbol{c}}_{\boldsymbol{N}} := \boldsymbol{c}_{\boldsymbol{N}}^{\mathrm{T}} - \boldsymbol{c}_{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{N}$.
- If *reduced cost* $\bar{\boldsymbol{c}}_{\boldsymbol{N}} \geq \boldsymbol{0}^{n-m}$ then $\boldsymbol{x}_{\boldsymbol{N}} = \boldsymbol{0}^{n-m}$ is optimal.

- If $\bar{\boldsymbol{c}}_{\boldsymbol{N}} \not\geq \boldsymbol{0}^{n-m}$ then $\exists j \in N$ with $\bar{c}_j < 0$. Then the current point $\boldsymbol{x}_{\boldsymbol{N}} = \boldsymbol{0}^{n-m}$ may be non-optimal.
- Generate a feasible descent direction!
- Choose one that leads to a neighboring extreme point!
- Swap one variable in B for one in N!
- Increase one variable in N from zero!
- Choose $\jmath^*$ to be among $\arg\text{minimum}_{j \in N}\ \bar{c}_j$.

The basis change

- In $\boldsymbol{x}_{\boldsymbol{N}}$-space: $\boldsymbol{p}_{\boldsymbol{N}} = \boldsymbol{e}_{\jmath^*}$ (unit vector)
- In $\boldsymbol{x}_{\boldsymbol{B}}$-space: $\boldsymbol{x}_{\boldsymbol{B}} = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_{\boldsymbol{N}} \Longrightarrow$ $\boldsymbol{p}_{\boldsymbol{B}} = -\boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{p}_{\boldsymbol{N}} = -\boldsymbol{B}^{-1}\boldsymbol{N}_{\jmath^*}$
- So, search direction in $\mathbb{R}^n$:

$$
\boldsymbol{p} = \begin{pmatrix} \boldsymbol{p}_{\boldsymbol{B}} \\ \boldsymbol{p}_{\boldsymbol{N}} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{B}^{-1}\boldsymbol{N}_{\jmath^*} \\ \boldsymbol{e}_{\jmath^*} \end{pmatrix}
$$

- Descent? Yes, because $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{p} = \bar{c}_{\jmath^*} < 0$!
- Feasible? Must check that $\boldsymbol{A}\boldsymbol{p} = \boldsymbol{0}^m$ and that $p_i \geq 0$ if $x_i = 0$, $i \in B$.

- (a) $\boldsymbol{Ap} = \boldsymbol{Bp_B} + \boldsymbol{Np_N} = -\boldsymbol{BB}^{-1}\boldsymbol{N}_{\jmath^*} + \boldsymbol{Ne}_{\jmath^*} = \boldsymbol{0}^m$
- (b) Suppose that $\boldsymbol{x_B} > \boldsymbol{0}^m$. Then, at least a small step in $\boldsymbol{p}$ keeps $\boldsymbol{x_B} \geq \boldsymbol{0}^m$. But if there is an $\imath^*$ with $(\boldsymbol{x_B})_{\imath^*} = 0$ *and* $(\boldsymbol{p_B})_{\imath^*} < 0$ then it is not a feasible direction.
- Must then perform a basis change without moving! A *degenerate basis change*: swap $x_{\jmath^*}$ for $x_{\imath^*}$ in the basis.
- Otherwise (and normally), we utilize the unit direction.
- Line search? Linear objective; move the maximum step!
- Maximum step: If $\boldsymbol{p_B} \geq \boldsymbol{0}^m$ there is no maximum step! We have found an extreme direction $\boldsymbol{p}$ along which $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ tends to $-\infty$! *Unbounded solution.*

- Otherwise: Some basic variables will reach zero eventually. Choose as the outgoing variable a variable $i \in B$ with minimum in

$$\underset{i \in B}{\text{minimum}} \left\{ \frac{(\boldsymbol{B}^{-1}\boldsymbol{b})_i}{(\boldsymbol{B}^{-1}\boldsymbol{N}_{\jmath^*})_i} \,\middle|\, (\boldsymbol{B}^{-1}\boldsymbol{N}_{\jmath^*})_i > 0 \right\}$$

- Done. In the basis, replace $\imath^*$ by $\jmath^*$! Go back to the pricing step.

Computational notes—how do we do all of this?

- Given basis matrix $\boldsymbol{B}$, solve

$$\boldsymbol{Bx_B} = \boldsymbol{b}.$$

- Gives us BFS: $\boldsymbol{x_B} = \boldsymbol{B}^{-1}\boldsymbol{b}$.
- Pricing step: (a) Solve

$$\boldsymbol{B}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c_B}.$$

- (b) Calculate $\bar{\boldsymbol{c}}_{\boldsymbol{N}}^{\mathrm{T}} = \boldsymbol{c}_{\boldsymbol{N}}^{\mathrm{T}} - \boldsymbol{y}^{\mathrm{T}}\boldsymbol{N}$, the reduced cost vector.
- Choose the incoming variable, $x_{\jmath^*}$.

- Outgoing variable: Solve

$$\boldsymbol{Bp_B} = -\boldsymbol{N}_{\jmath^*}.$$

- The quotient rule for $(\boldsymbol{B}^{-1}\boldsymbol{b})_i/(-\boldsymbol{p_B})_i$ gives the outgoing variable, $x_{\imath^*}$.
- Note: Three similar linear systems in $\boldsymbol{B}$! LU factorization + three triangular substitutions!
- Factorizations can be updated after basis change rather than done from scratch.
- LP solvers like Cplex and XPRESS-MP have excellent numerical solvers for linear systems.
- Linear systems the bulk of the work in solving an LP.

Convergence

- Theorem 10.10: *If all of the basic feasible solutions are non-degenerate, then the Simplex algorithm terminates after a finite number of iterations.*
- Rough argument: Non-degeneracy implies that the step length is > 0; hence, we cannot return to an old BFS once we have left it. There are finitely many BFSs. □

- Degeneracy: Can actually lead to cycling—the same sequence of BFSs is returned to indefinitely!
- Remedy: Change the incoming/outgoing criteria! Bland's rule: Sort variables according to some index ordering. Take the first possible index in the list. Incoming variable first in the list with the right sign of the reduced cost; outgoing variable the first in the list among the minima in the quotient rule.

Initial BFS: Phase I of the Simplex method

- If a starting BFS cannot be found, do the following.
- Suppose $\boldsymbol{b} \geq \boldsymbol{0}^m$. Introduce *artificial variables* a_i in every row (or rows without a unit column).
- Solve the following Phase-I problem:

$$\begin{aligned} \text{minimize} \quad w = \quad & (\boldsymbol{1}^m)^{\mathrm{T}}\boldsymbol{a} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{I}^m\boldsymbol{a} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \\ & \boldsymbol{a} \geq \boldsymbol{0}^m. \end{aligned}$$

- Possible cases: (a) $w^* = 0$, meaning that $\boldsymbol{a}^* = \boldsymbol{0}^m$ must hold. There is then a BFS in the *original* problem.

- Start Phase-II, to solve the original problem, starting from this BFS.
- (b) $w^* > 0$. The optimal basis then has some $a_i^* > 0$; due to the objective function construction, there exists no BFS in the original problem. The problem is then infeasible!
- What to do then? Modelling errors? Can be detected from the optimal solution. In fact some LP problems are pure feasibility problems.