

Lecture 11: Integer programming

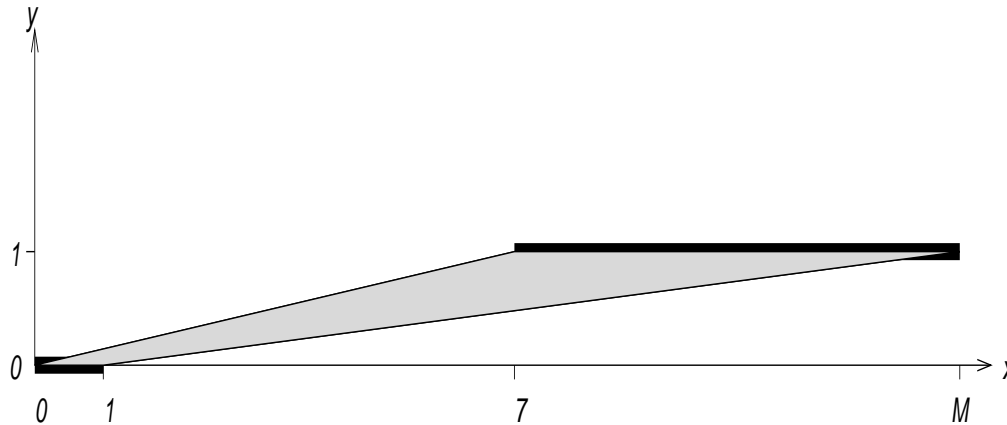
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22 February 2005

When are integer models needed?

- Products or raw materials are indivisible
- Logical constraints: “if A then B ”; “ A or B ”
- Fixed costs
- Combinatorics (sequencing, allocation)
- On/off-decision to buy, invest, hire, generate electricity,
...

Either $0 \leq x \leq 1$ or $x \geq 7$



Let $M \gg 1$: $x \leq 1 + My$, $x \geq 7y$, $y \in \{0, 1\}$

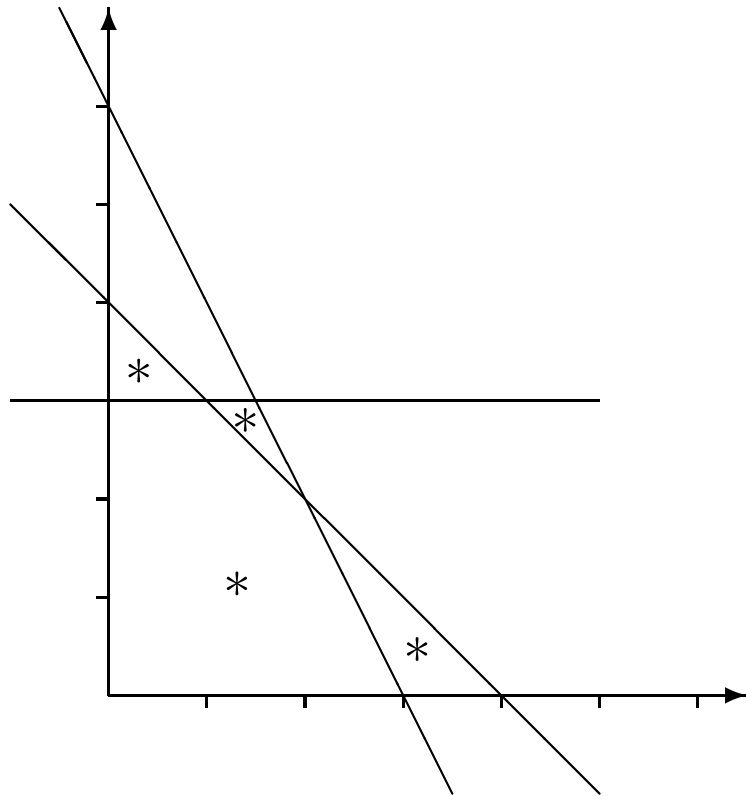
Variable x may only take the values 2, 45, 78 & 107

$$x = 2y_1 + 45y_2 + 78y_3 + 107y_4$$

$$y_1 + y_2 + y_3 + y_4 = 1$$

$$y_1, y_2, y_3, y_4 \in \{0, 1\}$$

At least 2 of 3 constraints must be fulfilled



* = feasible regions

$$M \geq 2$$

$$x_1 + x_2 \leq 4 \quad (1)$$

$$2x_1 + x_2 \leq 6 \quad (2)$$

$$x_2 \leq 3 \quad (3)$$

$$\text{and } x_1, x_2 \geq 0$$

$$x_1 + x_2 \leq 4 + M(1 - y_1) \quad (1)$$

$$2x_1 + x_2 \leq 6 + M(1 - y_2) \quad (2)$$

$$x_2 \leq 3 + M(1 - y_3) \quad (3)$$

$$y_1 + y_2 + y_3 \geq 2$$

$$y_1, y_2, y_3 \in \{0, 1\}$$

$$\text{and } x_1, x_2 \geq 0$$

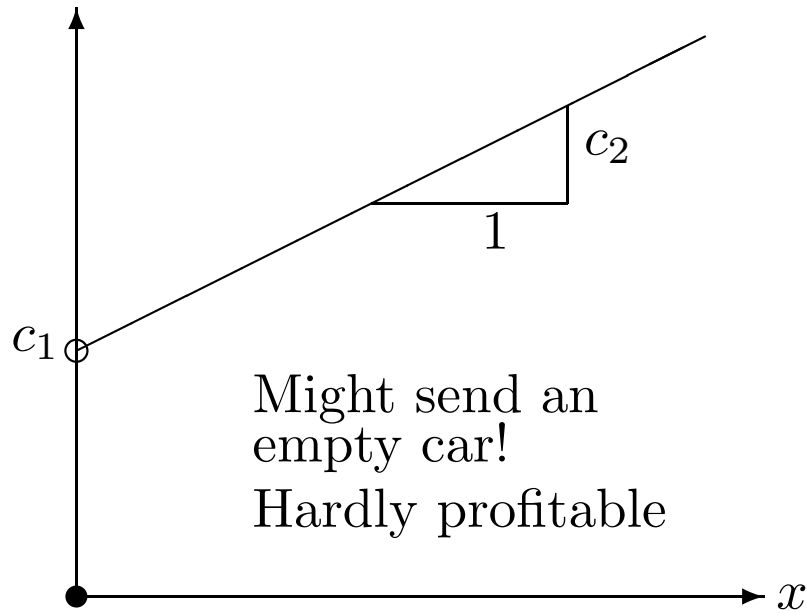
Fixed costs

x = the amount of a certain product to be sent

If $x > 0$ then the initial cost c_1 (e.g. car hire) is generated

Variable cost c_2 per unit sent

$$\text{Total cost: } f(x) = \begin{cases} 0 & \text{if } x = 0 \quad \boxed{\text{effect}} \\ c_1 + c_2 \cdot x & \text{if } x > 0 \quad \boxed{\text{wanted!}} \end{cases}$$



Let M = car capacity

$$y = \begin{cases} 1 & \text{if } x > 0 \quad \boxed{\text{effect}} \\ 0 & \text{if } x = 0 \quad \boxed{\text{wanted!}} \end{cases}$$

$$f(x, y) = c_1 \cdot y + c_2 \cdot x$$

$$x \leq M \cdot y \quad \boxed{\text{linear 0/1 model!}}$$

$$x \geq 0, \quad y \in \{0, 1\}$$

Other applications of integer optimization

- Facility location (new hospitals, shopping centers, etc.)
- Scheduling (on machines, personnel, projects, schools)
- Logistics (material- and warehouse control)
- Distribution (transportation of goods, buses for disabled persons)
- Production planning
- Telecommunication (network design, frequency allocation)
- VLSI design

The combinatorial explosion

Assign n persons to carry out n jobs. # feasible solutions: $n!$

Assume that a feasible solution is evaluated in 10^{-9} seconds

n	2	5	8	10	100
$n!$	2	120	$4.0 \cdot 10^4$	$3.6 \cdot 10^6$	$9.3 \cdot 10^{157}$
[time]	10^{-8} s	10^{-6} s	10^{-4} s	10^{-2} s	10^{142} yrs

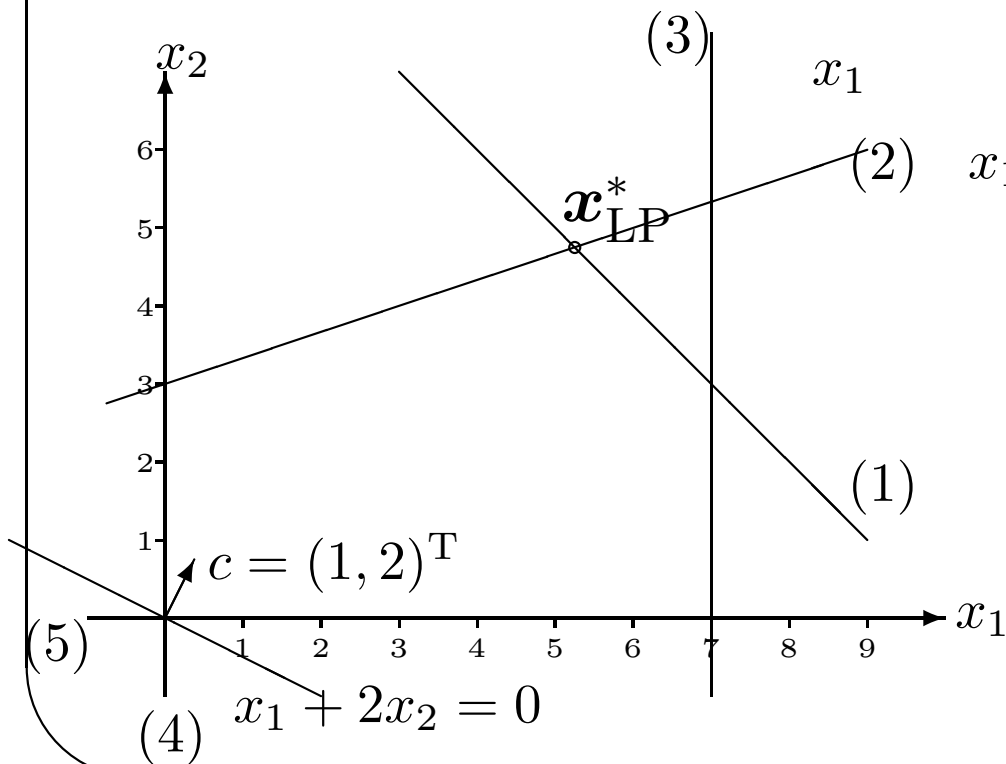
Complete enumeration of all solutions is **not** an efficient algorithm!

An algorithm exists that solves this problem in time $\mathcal{O}(n^4) \propto n^4$

n	2	5	8	10	100	1000
n^4	16	625	$4.1 \cdot 10^3$	10^4	10^8	10^{12}
[time]	10^{-7} s	10^{-6} s	10^{-5} s	10^{-5} s	10^{-1} s	17 min

Linear continuous optimization model

$$\begin{aligned} \max \quad z_{\text{LP}} &= x_1 + 2x_2 \\ \text{s.t.} \quad &x_1 + x_2 \leq 10 \quad (1) \\ &-x_1 + 3x_2 \leq 9 \quad (2) \\ &x_1 \leq 7 \quad (3) \\ &x_1, x_2 \geq 0 \quad (4, 5) \end{aligned}$$



$$x_{\text{LP}}^* = \begin{pmatrix} 21/4 \\ 19/4 \end{pmatrix}$$

$$z_{\text{LP}}^* = 14\frac{3}{4}$$

Linear integer optimization model

$$\max z_{\text{IP}} = x_1 + 2x_2$$

$$\text{s.t.} \quad x_1 + x_2 \leq 10 \quad (1)$$

$$-x_1 + 3x_2 \leq 9 \quad (2)$$

• = feasible
integer points

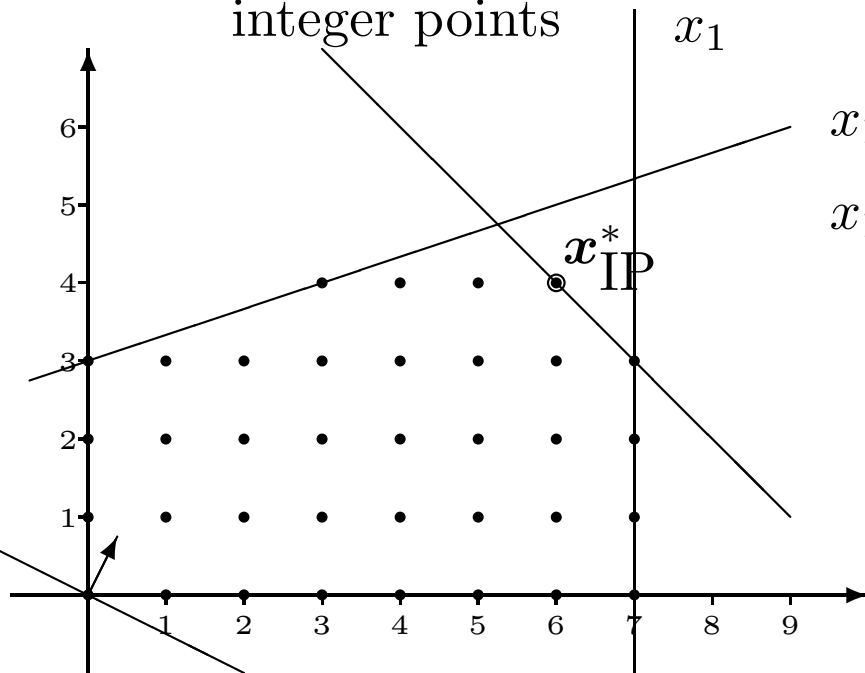
$$x_1 \leq 7 \quad (3)$$

$$x_1, x_2 \geq 0 \quad (4, 5)$$

x_1, x_2 integer

$$\mathbf{x}_{\text{IP}}^* = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$z_{\text{IP}}^* = 14 < z_{\text{LP}}^*$$

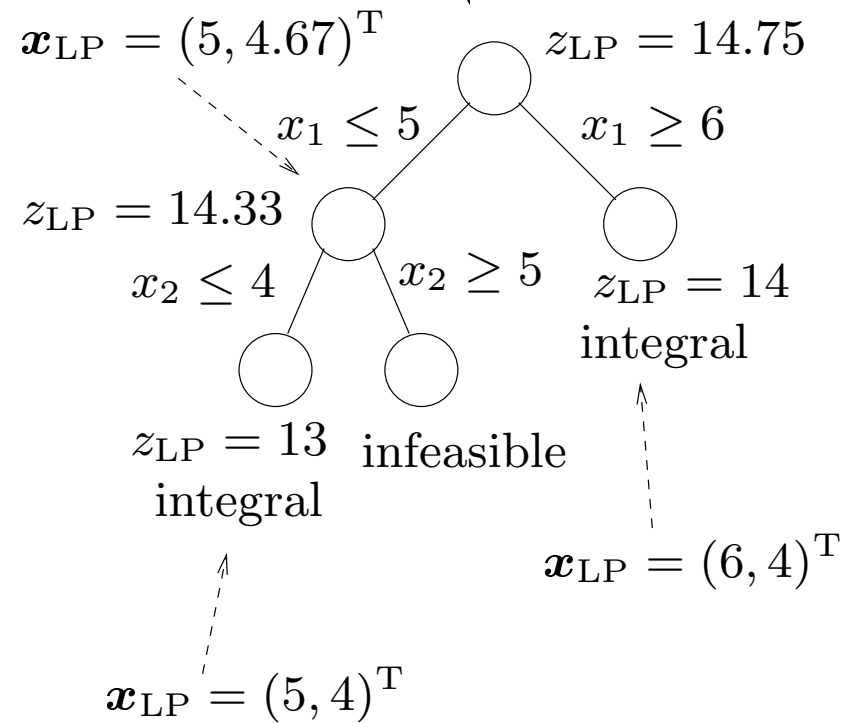
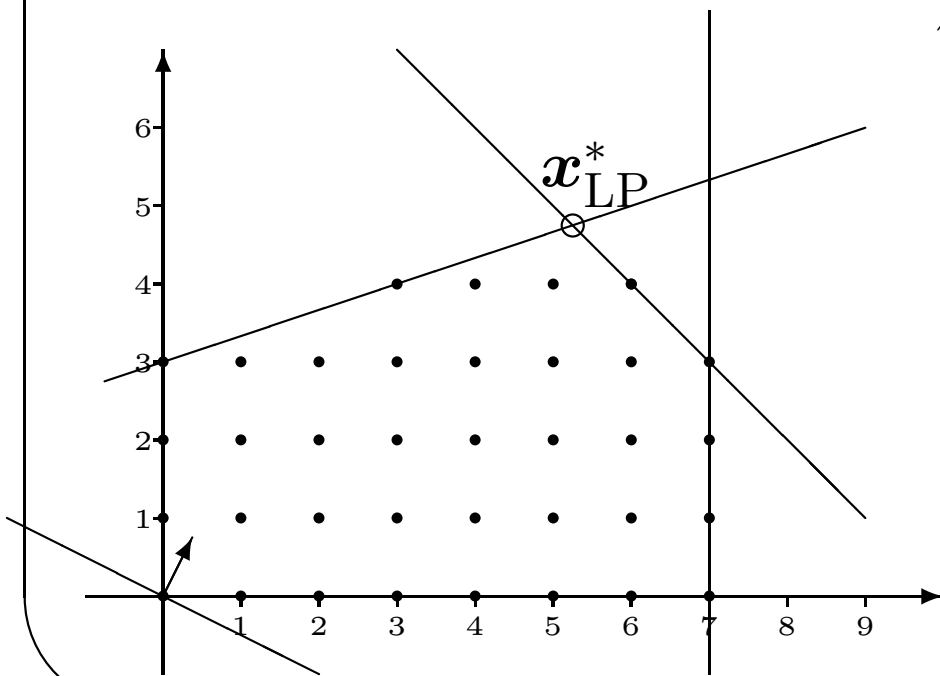


Classic methods

- *Branch-and-Bound*: relaxation plus divide-and-conquer
- *Cutting plane method*: relaxation plus generations of constraint that cut off infeasible (e.g., non-integer) points generated
- “Relaxation” can be the continuous or Lagrangian one
- *Lagrangian optimization*: Lagrangian relaxation plus multiplier optimization
- These methods are often combined (e.g., cutting planes added at nodes in B & B tree: Branch & Cut)

The branch-and-bound-algorithm

Relax integrality constraints \Rightarrow linear program $\Rightarrow \mathbf{x}_{LP}^* = (5.25, 4.75)^T$



The complexity of integer optimization, I: Mexico

- The Mexico LP has 113 variables and 84 linear constraints. Solution by a *slow* (333 MHz Unix) computer: 0.01 s.
- We create an integer programming (IP) variant: add a fixed cost for using a railway link for the raw material transport. 78 binary (0/1) variables.
- Cplex uses B & B, in which to the continuous relaxation is added integer requirements on some of the binary variables that received a fractional value in the LP solution. (Note: x binary here \implies variable values fixed at 0 or 1)

- Solution times:

Fixed cost 100 \implies 20 s.

18,000 B & B nodes

60,000 simplex iterations

300 \implies 3 min.

208,000 B & B nodes

650,000 simplex iterations

- There are $2^{78} \approx 0.3 \cdot 10^{24}$ possible combinations. B & B is good at *implicitly* enumerating them all
- Higher fixed cost \implies more difficult problem. Why?
Continuous relaxation worse approximation!

The complexity of integer optimization, II: The knapsack problem

- Knapsack problem: maximize value of a finite number of items put in a knapsack of a given capacity
- Each variable has a value and weight per unit
- AMPL model:

```
var x1..5 integer, >=0;
```

```
maximize ka:213*x[1]-1928*x[2]-11111*x[3]-2345*x[4]+9123*x[5];
```

```
subject to c1:
```

```
12223*x[1]+12224*x[2]+36674*x[3]+61119*x[4]+85569*x[5] =
```

```
89643482;
```

- Often binary; here, general integer variables

- LP relaxation trivial: sort variables in descending order of c_j/a_j ; take the best one
- Result: After 10 minutes in CPLEX: 8 Million B & B nodes; no feasible solution

Cutting plane methods

- Goal: generate the convex hull of the feasible integer vectors
- Result: Can solve the IP by solving the LP relaxation over this convex hull
- Compare IP example: one extra linear constraint defines the entire convex hull! ($x_2 \leq 4$)
- Means: Relax problem (e.g., continuous relaxation); Solve. If infeasible solution, generate constraint to the relaxation that cuts off that vector but no feasible vectors. Repeat.
- Constraint generation called a *separation oracle*

- Niclas Ph.D. thesis project deals with a large-scale mixed-integer optimization problem, solvable by special cutting plane methods
- Important to generate constraints that cut off vectors in a high dimension, especially so-called facets—faces of a polyhedron with the maximal dimension
- Niclas has established a class of such facets for “his” (convex hull) polyhedron

The Philips example—TSP solved heuristically

- Let c_{ij} denote the distance between cities i and j , with

$\{i, j\} \subset \mathcal{N}$ — set of nodes

$(i, j) \in \mathcal{L}$ — set of links

- Links (i, j) and (j, i) the same; direction plays no role

- $x_{ij} = \begin{cases} 1, & \text{if link } (i, j) \text{ is part of the TSP tour,} \\ 0, & \text{otherwise} \end{cases}$

- The Traveling Salesman Problem (TSP):

$$\text{minimize} \quad \sum_{(i,j) \in \mathcal{L}} c_{ij} x_{ij}$$

$$\text{subject to} \quad \sum_{(i,j) \in \mathcal{L}: \{i,j\} \subset \mathcal{S}} x_{ij} \leq |\mathcal{S}| - 1, \quad \mathcal{S} \subset \mathcal{N}, \quad (1)$$

$$\sum_{(i,j) \in \mathcal{L}} x_{ij} = n, \quad (2)$$

$$\sum_{i \in \mathcal{N}: (i,j) \in \mathcal{L}} x_{ij} = 2, \quad j \in \mathcal{N}, \quad (3)$$

$$x_{ij} \in \{0, 1\}, \quad (i, j) \in \mathcal{L}$$

Interpretations

- Constraint (1) implies that there can be no *sub-tours*, that is, a tour where fewer than n cities are visited (that is, if $\mathcal{S} \subset \mathcal{N}$ then there can be at most $|\mathcal{S}| - 1$ links between nodes in the set \mathcal{S} , where $|\mathcal{S}|$ is the cardinality–number of members of–the set \mathcal{S});
- Constraint (2) implies that in total n cities must be visited;
- Constraint (3) implies that each city is connected to two others, such that we make sure to arrive from one city and leave for the next.

Lagrangian relaxation

- TSP is NP-hard—no known polynomial algorithms exist
- Lagrangian relax (3) for all nodes except starting node
- Remaining problem: 1-MST—find the minimum spanning tree in the graph without the starting node and its connecting links; then, add the two cheapest links to connect the starting node
- Starting node $s \in \mathcal{N}$ and connected links assumed removed from the graph

- Objective function of the Lagrangian problem:

$$\begin{aligned}
 q(\boldsymbol{\lambda}) &= \underset{\boldsymbol{x}}{\text{minimum}} \sum_{(i,j) \in \mathcal{L}} c_{ij} x_{ij} + \sum_{j \in \mathcal{N}} \lambda_j \left(2 - \sum_{i \in \mathcal{N}: (i,j) \in \mathcal{L}} x_{ij} \right) \\
 &= 2 \sum_{j \in \mathcal{N}} \lambda_j + \underset{\boldsymbol{x}}{\text{minimum}} \sum_{(i,j) \in \mathcal{L}} (c_{ij} - \lambda_i - \lambda_j) x_{ij}
 \end{aligned}$$

- A high (low) value of the multiplier λ_j makes node j attractive (unattractive) in the 1-MST problem, and will therefore lead to more (less) links being attached to it
- Subgradient method for updating the multipliers

- Updating step:

$$\lambda_j := \lambda_j + \alpha \left(2 - \sum_{i \in \mathcal{N}: (i,j) \in \mathcal{L}} x_{ij} \right), \quad j \in \mathcal{N},$$

where $\alpha > 0$ is a step length

- Update means:

Current degree at node j :

$$\left\{ \begin{array}{l} > 2 \implies \lambda_j \downarrow \text{ (link cost } \uparrow \text{)} \\ = 2 \implies \lambda_j \leftrightarrow \text{ (link cost constant)} \\ < 2 \implies \lambda_j \uparrow \text{ (link cost } \downarrow \text{)} \end{array} \right.$$

- Link cost shifted upwards (downwards) if too many (too few) links connected to node j in the 1-MST.

Feasibility heuristic

- Adjusts Lagrangian solution \boldsymbol{x} such that the resulting vector is feasible
- Often a good thing to do when approaching the dual optimal solution— \boldsymbol{x} often then only mildly infeasible
- Identify path in 1-MST with many links; form a subgraph with the remaining nodes which is a path; connect the two
- Result: A Hamiltonian cycle (TSP tour)
- We then have both an upper bound (feasible point) and a lower bound (q) on the optimal value—a quality measure!

The Philips example

- Fixed number of subgradient iterations
- Feasibility heuristic used every K iterations ($K > 1$), starting at a late subgradient iteration
- Typical example: Optimal path length in the order of 2 meters; upper and lower bounds produced concluded that the relative error in the production plan is *less than 7 %*
- Also: increase in production by some 70 %