Lecture 2: Convexity

Convexity of sets

Let $S \subseteq \mathbb{R}^n$. The set S is *convex* if

$$\left.\begin{array}{l} \boldsymbol{x}^1, \boldsymbol{x}^2 \in S \\ \lambda \in (0,1) \end{array}\right\} \quad \Longrightarrow \quad \lambda \boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2 \in S.$$

A set S is convex if, from anywhere in S, all other points are "visible." (See Figure 1.)

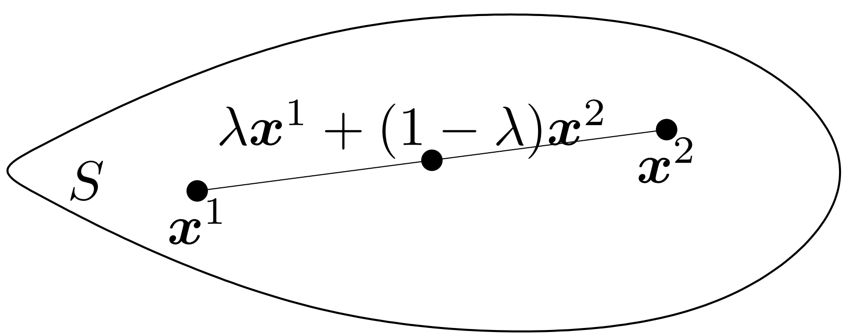

Figure 1: A convex set. (For the intermediate vector shown, the value of λ is $\approx 1/2$.)

Examples

- The empty set is a convex set.
- The set $\{ \boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\| \leq a \}$ is convex for every value of $a \in \mathbb{R}$.
- The set $\{ \boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\| = a \}$ is non-convex for every $a > 0$.
- The set $\{0, 1, 2\}$ is non-convex.

Two non-convex sets are shown in Figure 2.

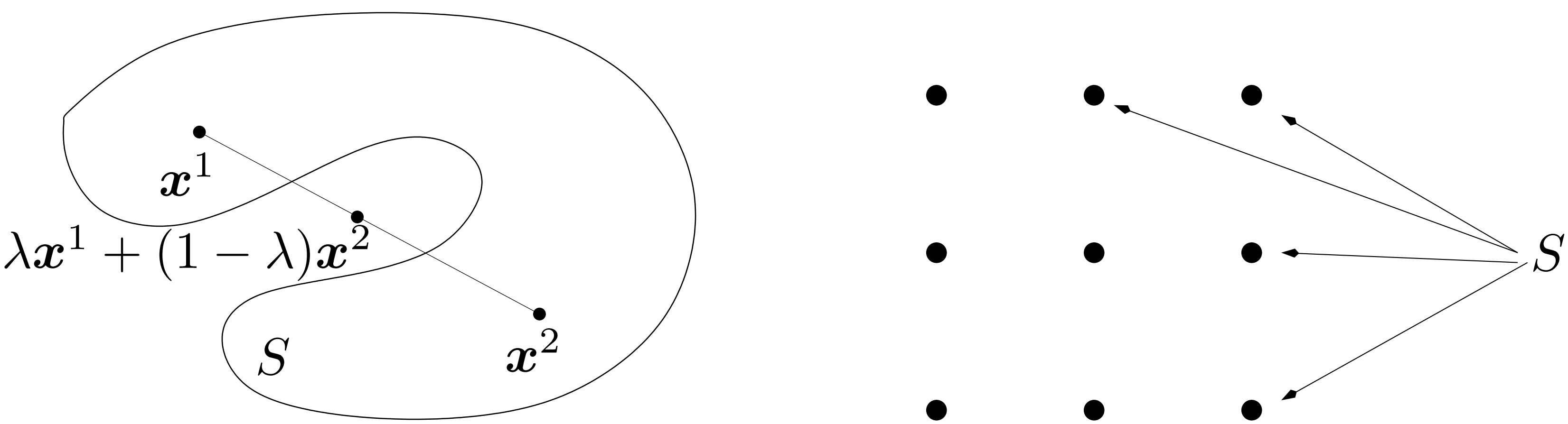

Figure 2: Two non-convex sets.

Intersections of convex sets

Suppose that S_k, $k \in \mathcal{K}$, is any collection of convex sets. Then, the intersection $\cap_{k \in \mathcal{K}} S_k$ is a convex set.

Proof.

Convex and affine hulls

The *affine hull* of a finite set $V = \{\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k\} \subset \mathbb{R}^n$ is the set

$$\operatorname{aff} V := \left\{ \lambda_1 \boldsymbol{v}^1 + \cdots + \lambda_k \boldsymbol{v}^k \;\middle|\; \lambda_1, \ldots, \lambda_k \in \mathbb{R};\; \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

The *convex hull* of a finite set $V = \{\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k\} \subset \mathbb{R}^n$ is the set

$$\operatorname{conv} V := \left\{ \lambda_1 \boldsymbol{v}^1 + \cdots + \lambda_k \boldsymbol{v}^k \;\middle|\; \lambda_1, \ldots, \lambda_k \geq 0;\; \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

The sets are defined by all possible *affine (convex) combinations* of the k points.

Examples

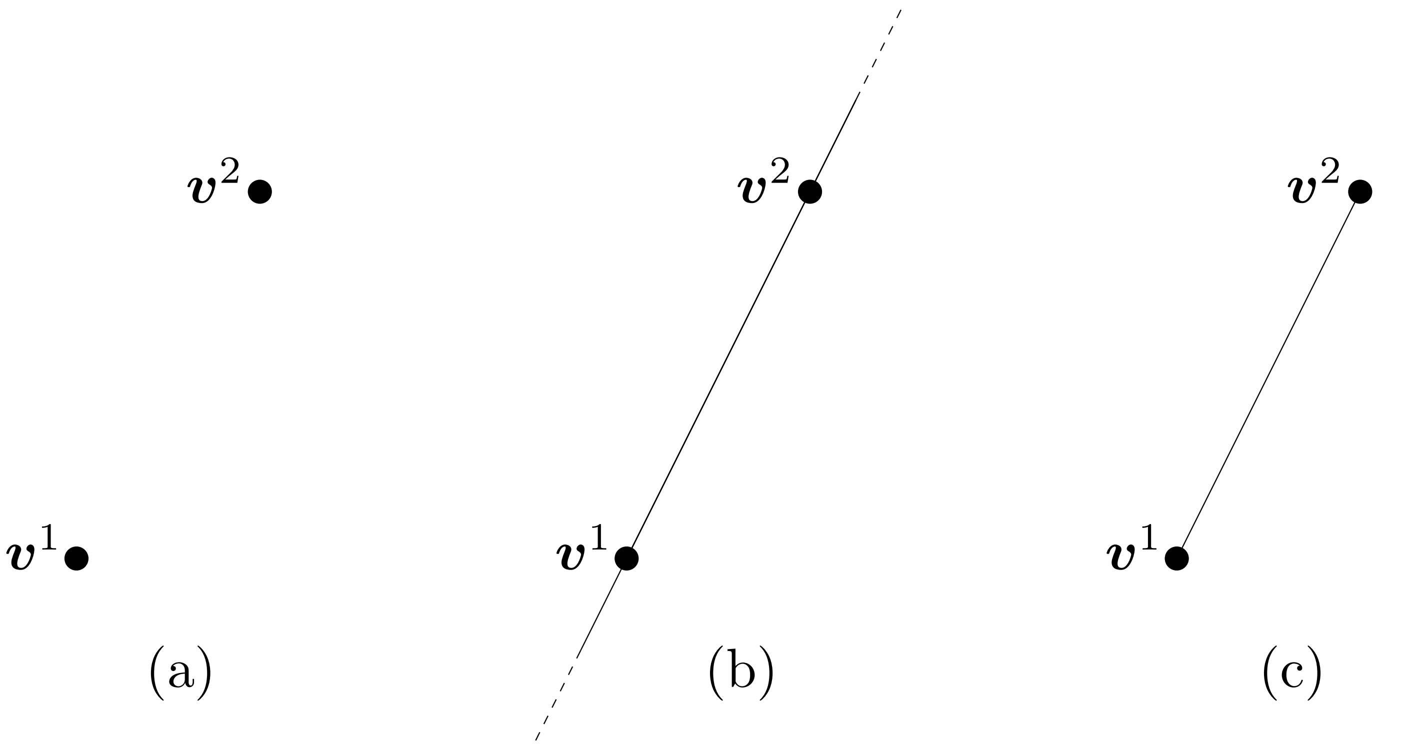

Figure 3: (a) The set V. (b) The set $\operatorname{aff} V$. (c) The set $\operatorname{conv} V$.

Carathéodory's Theorem

- The convex hull of $V \subset \mathbb{R}^n$ is the smallest convex set containing V.
- Let $V \subseteq \mathbb{R}^n$. Then, $\operatorname{conv} V$ is the set of all convex combinations of points of V.
- Every point of the convex hull of a set can be written as a convex combination of points from the set. How many do we need?
- [Car.:] *Let* $\boldsymbol{x} \in \operatorname{conv} V$*, where* $V \subseteq \mathbb{R}^n$*. Then* $\boldsymbol{x}$ *can be expressed as a convex combination of* $n+1$ *or fewer points of* V*.*
- Proof by contradiction: if more than $n+1$ points are needed then these points must be affinely dependent $\Longrightarrow$ can remove at least one such point. Etcetera.

Polytope

- A subset P of $\mathbb{R}^n$ is a *polytope* if it is the convex hull of finitely many points in $\mathbb{R}^n$.
- The set shown in Figure 4 is a polytope.

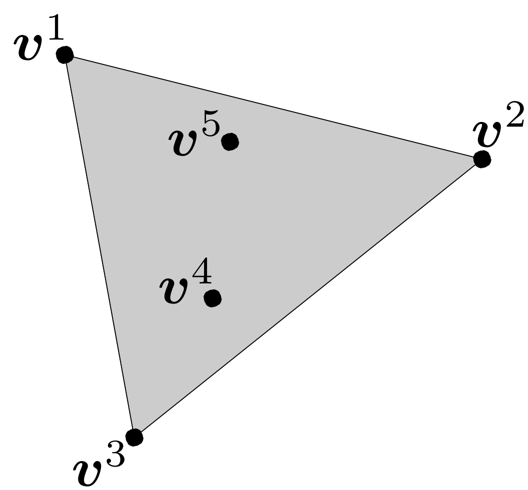

Figure 4: The convex hull of five points in $\mathbb{R}^2$.

- A cube and a tetrahedron are polytopes in $\mathbb{R}^3$.

Extreme points

- A point $\boldsymbol{v}$ of a convex set P is called an *extreme point* if whenever $\boldsymbol{v} = \lambda \boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2$, where $\boldsymbol{x}^1, \boldsymbol{x}^2 \in P$ and $\lambda \in (0,1)$, then $\boldsymbol{v} = \boldsymbol{x}^1 = \boldsymbol{x}^2$.
- Examples: The set shown in Figure 3(c) has the extreme points $\boldsymbol{v}^1$ and $\boldsymbol{v}^2$. The set shown in Figure 4 has the extreme points $\boldsymbol{v}^1$, $\boldsymbol{v}^2$, and $\boldsymbol{v}^3$. The set shown in Figure 3(b) does not have any extreme points.
- Let P be the polytope $\operatorname{conv} V$, where $V = \{\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k\} \subset \mathbb{R}^n$. Then P is equal to the convex hull of its extreme points.

Polyhedra

- A subset P of $\mathbb{R}^n$ is a *polyhedron* if there exist a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and a vector $\boldsymbol{b} \in \mathbb{R}^n$ such that

$$P = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \,\}.$$

- $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \iff \boldsymbol{a}_i\boldsymbol{x} \leq b_i$ for all i ($\boldsymbol{a}_i$ is row i of $\boldsymbol{A}$).
- Intersection of half-spaces. [*Hyperplane:* $\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{a}_i\boldsymbol{x} = b_i \,\}$.]
- Examples: (a) Figure 5 shows the bounded polyhedron $P = \{\, \boldsymbol{x} \in \mathbb{R}^2 \mid x_1 \geq 2;\; x_1 + x_2 \leq 6;\; 2x_1 - x_2 \leq 4 \,\}$.
- (b) The unbounded polyhedron $P = \{\, \boldsymbol{x} \in \mathbb{R}^2 \mid x_1 + x_2 \geq 2;\; x_1 - x_2 \leq 2;\; 3x_1 - x_2 \geq 0 \,\}$ is shown in Figure 6.

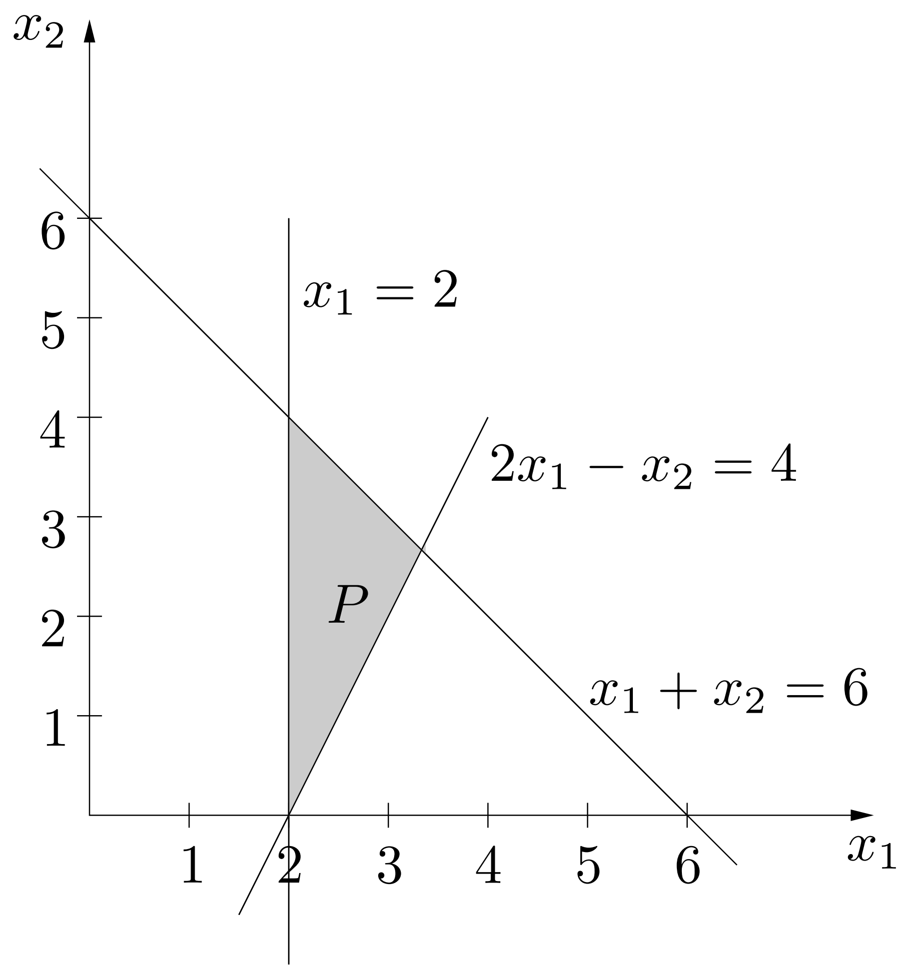

Figure 5: Illustration of the bounded polyhedron $P = \{\, \boldsymbol{x} \in \mathbb{R}^2 \mid x_1 \geq 2;\; x_1 + x_2 \leq 6;\; 2x_1 - x_2 \leq 4 \,\}$.

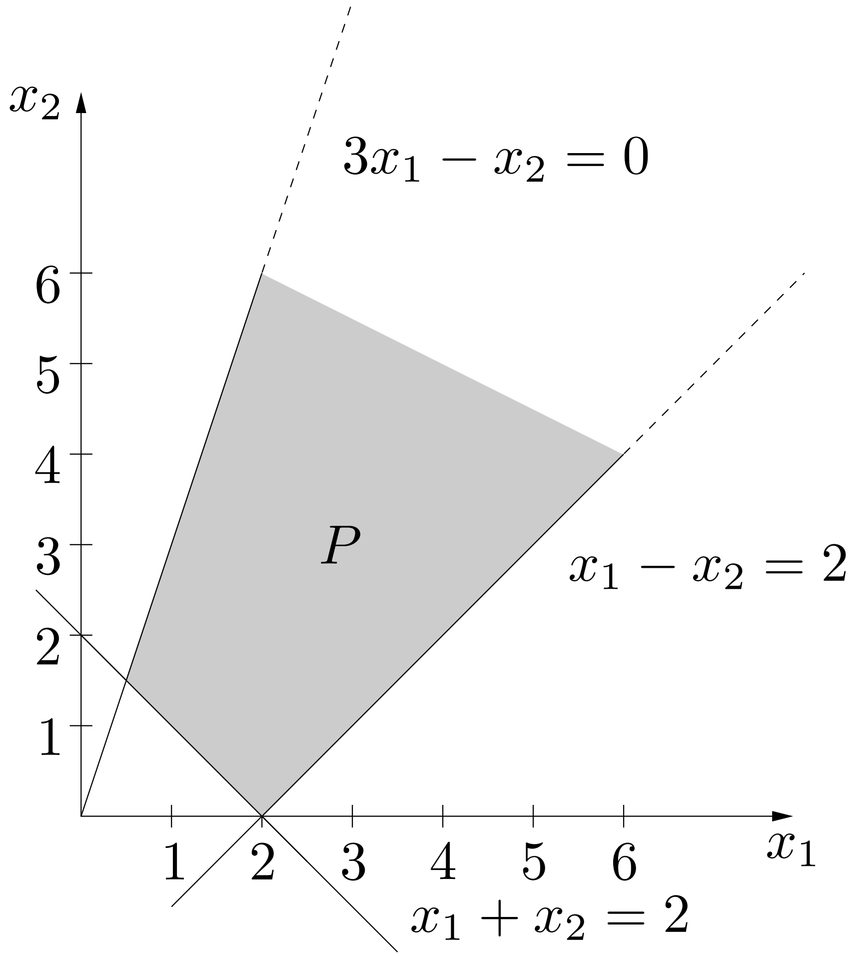

Figure 6: Illustration of the unbounded polyhedron $P = \{\, \boldsymbol{x} \in \mathbb{R}^2 \mid x_1 + x_2 \geq 2;\ x_1 - x_2 \leq 2;\ 3x_1 - x_2 \geq 0 \,\}$.

Algebraic characterizations of extreme points

- *Let* $\tilde{\boldsymbol{x}} \in P = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \,\}$, *where* $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ *with* $\operatorname{rank} \boldsymbol{A} = n$ *and* $\boldsymbol{b} \in \mathbb{R}^m$*. Further, let* $\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}}$ *be the equality subsystem of* $\boldsymbol{A}\tilde{\boldsymbol{x}} \le \boldsymbol{b}$*. Then* $\tilde{\boldsymbol{x}}$ *is an extreme point of* P *if and only if* $\operatorname{rank} \tilde{\boldsymbol{A}} = n$*.*

- Of great importance in Linear Programming: $\boldsymbol{A}$ then always has full rank! Hence, can solve special subsystem of linear equalities to obtain an extreme point.

- Corollary: The number of extreme points of P is finite.

- Corollary: Since the number of extreme points is finite, the convex hull of the extreme points of a polyhedron is a polytope.

- Consequence: Algorithm for linear programming!

Cones

- A subset C of $\mathbb{R}^n$ is a *cone* if $\lambda \boldsymbol{x} \in C$ whenever $\boldsymbol{x} \in C$ and $\lambda > 0$.
- Example: Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$. The set $\{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \leq \mathbf{0}^m \,\}$ is a cone.
- Figure 7(a) illustrates a convex cone and Figure 7(b) illustrates a non-convex cone in $\mathbb{R}^2$.

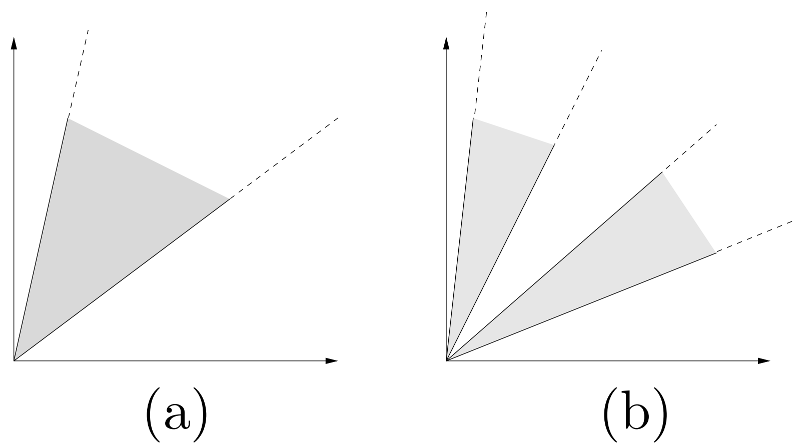

Figure 7: (a) A convex cone in $\mathbb{R}^2$. (b) A non-convex cone in $\mathbb{R}^2$.

Representation Theorem

- *Let* $Q = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}\,\}$, P *be the convex hull of the extreme points of* Q, *and* $C := \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{0}^m \,\}$. *If* $\operatorname{rank} \boldsymbol{A} = n$ *then*
  $Q = P + C = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} = \boldsymbol{u} + \boldsymbol{v}$ *for some* $\boldsymbol{u} \in P$ *and* $\boldsymbol{v} \in C \,\}$.
  In other words, every polyhedron (that has at least one extreme point) is the direct sum of a polytope and a polyhedral cone.
- Proof by induction on the rank of the subsystem matrix $\tilde{\boldsymbol{A}}$.
- Central in Linear Programming. Can be used to establish: *Optimal solutions to LP problems are found at extreme points*!

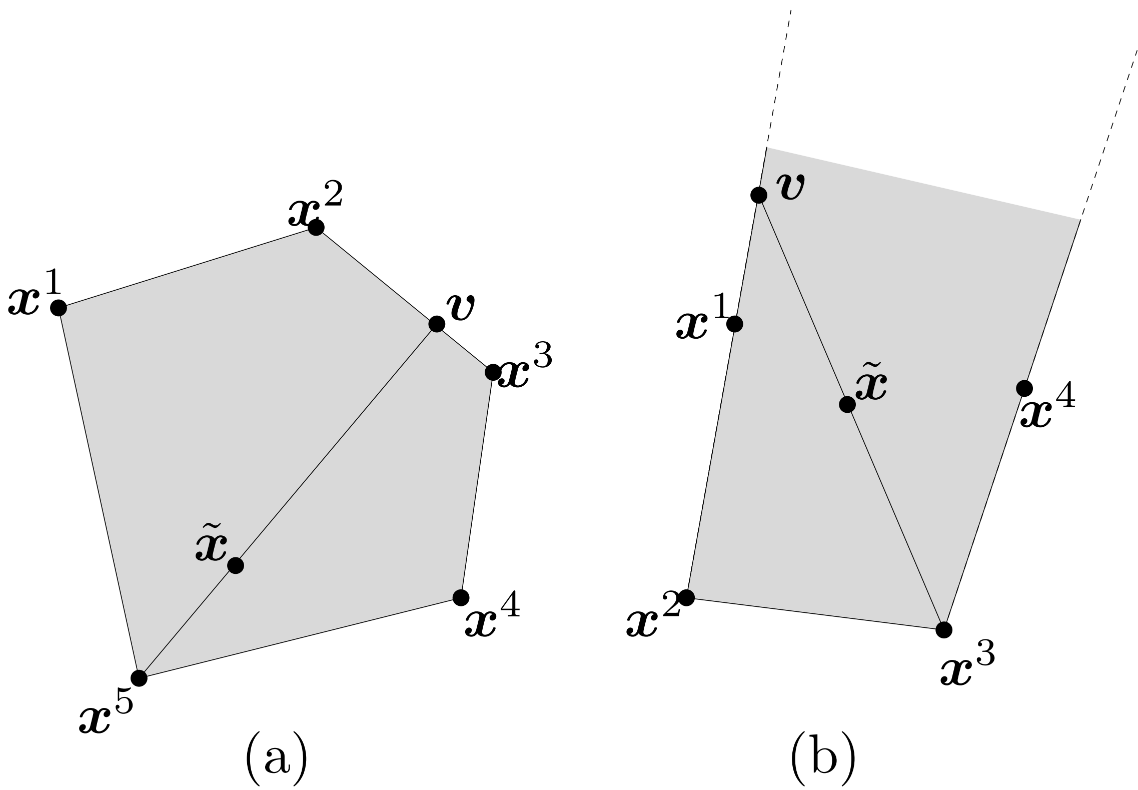

Figure 8: Illustration of the Representation Theorem (a) in the bounded case, and (b) in the unbounded case.

Separation Theorem

- "If a point $\boldsymbol{y}$ does not lie in a closed and convex set C, then there exists a hyperplane that separates $\boldsymbol{y}$ from C."
- *Suppose that the set $C \subseteq \mathbb{R}^n$ is closed and convex, and that the point $\boldsymbol{y}$ does not lie in C. Then there exist $\alpha \in \mathbb{R}$ and $\boldsymbol{\pi} \neq \mathbf{0}^n$ such that $\boldsymbol{\pi}^{\mathrm{T}}\boldsymbol{y} > \alpha$ and $\boldsymbol{\pi}^{\mathrm{T}}\boldsymbol{x} \leq \alpha$ for all $\boldsymbol{x} \in C$.*
- Proof later—requires existence and optimality conditions.
- Consequence: *A set P is a polytope if and only if it is a bounded polyhedron.* [$\Longleftarrow$ trivial; $\Longrightarrow$ constructive.]
- A *finitely generated* cone has the form

$$\operatorname{cone}\{\boldsymbol{v}^1, \ldots, \boldsymbol{v}^m\} := \{\, \lambda_1\boldsymbol{v}^1 + \cdots + \lambda_m\boldsymbol{v}^m \mid \lambda_1, \ldots, \lambda_m \geq 0 \,\}.$$

- *A convex cone is finitely generated iff it is polyhedral.*

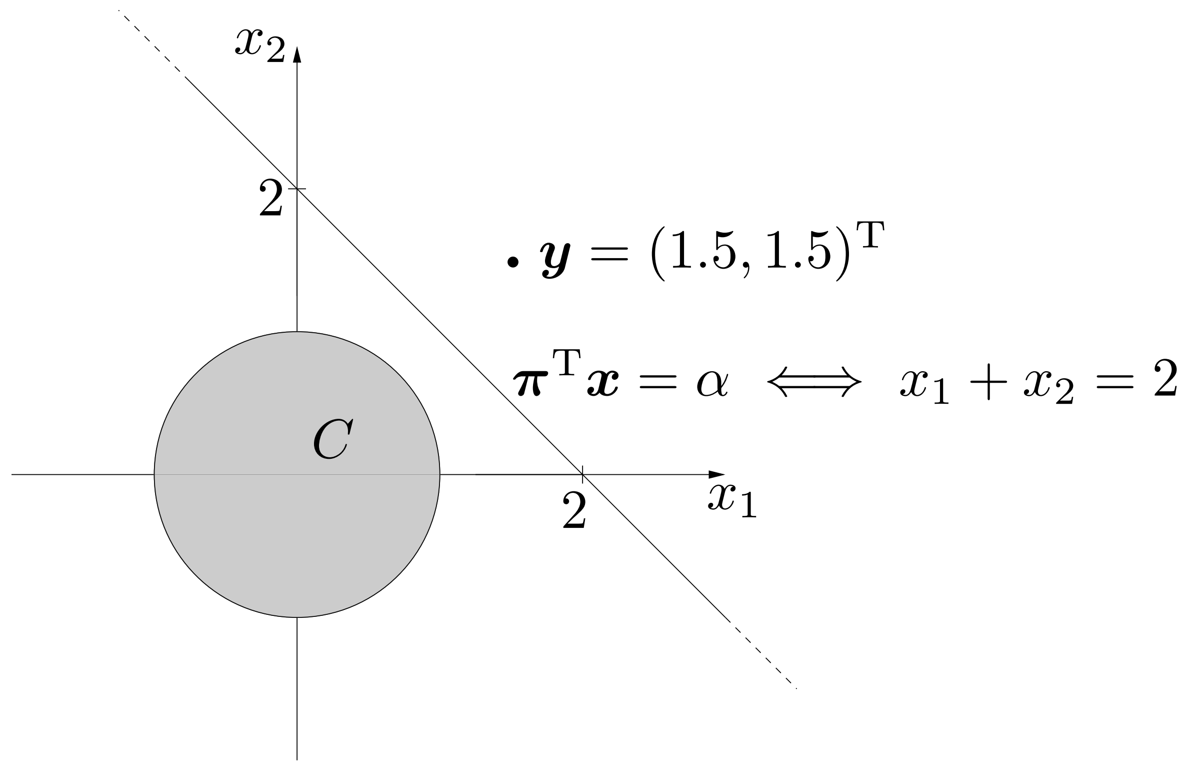

Figure 9: Illustration of the Separation Theorem: the unit disk is separated from $\boldsymbol{y}$ by the line $\{\, \boldsymbol{x} \in \mathbb{R}^2 \mid x_1 + x_2 = 2 \,\}$.

Farkas' Lemma

- *Let* $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ *and* $\boldsymbol{b} \in \mathbb{R}^m$*. Then, exactly one of the systems*

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b}, \\ \boldsymbol{x} &\geq \mathbf{0}^n, \end{aligned} \tag{I}$$

  and

$$\begin{aligned} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\pi} &\leq \mathbf{0}^n, \\ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{\pi} &> 0, \end{aligned} \tag{II}$$

  has a feasible solution, and the other system is inconsistent.

- Farkas' Lemma has many forms. "Theorems of the alternative."
- Crucial for LP theory and optimality conditions.
- Simple proof later!

Convexity of functions

- Suppose that $S \subseteq \mathbb{R}^n$ is convex. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is *convex at* $\bar{\boldsymbol{x}} \in S$ if

$$\left.\begin{array}{l} \boldsymbol{x} \in S \\ \lambda \in (0,1) \end{array}\right\} \Longrightarrow f(\lambda\bar{\boldsymbol{x}} + (1-\lambda)\boldsymbol{x}) \leq \lambda f(\bar{\boldsymbol{x}}) + (1-\lambda) f(\boldsymbol{x}).$$

- The function f is *convex on* S if it is convex at every $\bar{\boldsymbol{x}} \in S$.
- The function f is *strictly convex on* S if $<$ holds in place of $\leq$ above for every $\boldsymbol{x} \neq \bar{\boldsymbol{x}}$.
- A convex function is such that a linear interpolation never is lower than the function itself. For a strictly convex function the linear interpolation lies above the function.
- (Strict) concavity of $f \iff$ (strict) convexity of $-f$.

- Figure 10 illustrates a convex function.

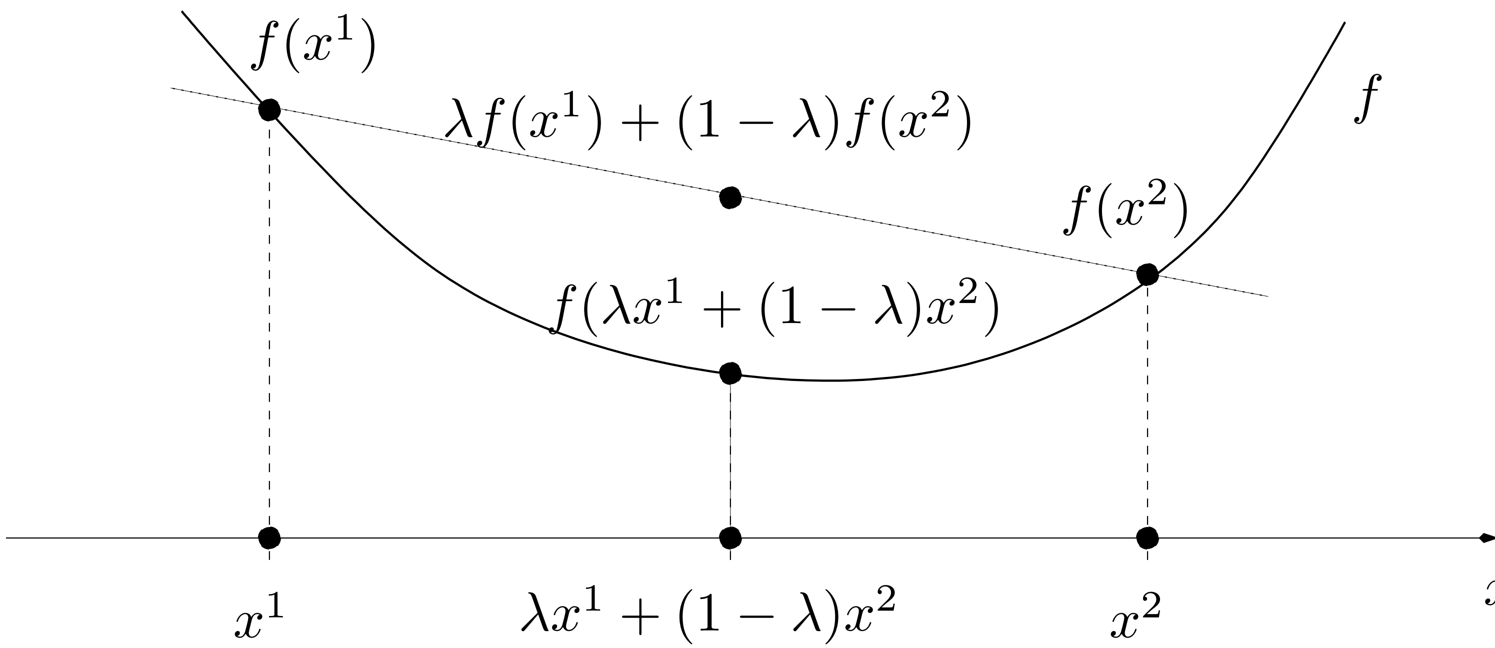

Figure 10: A convex function.

- The function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(\boldsymbol{x}) := \|\boldsymbol{x}\|$ is convex on $\mathbb{R}^n$; $f(\boldsymbol{x}) := \|\boldsymbol{x}\|^2$ is strictly convex in $\mathbb{R}^n$.

- Let $\boldsymbol{c} \in \mathbb{R}^n$. The linear function $\boldsymbol{x} \mapsto f(\boldsymbol{x}) := \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = \sum_{j=1}^{n} c_j x_j$ is both convex and concave on $\mathbb{R}^n$.
- Figure 11 illustrates a non-convex function.

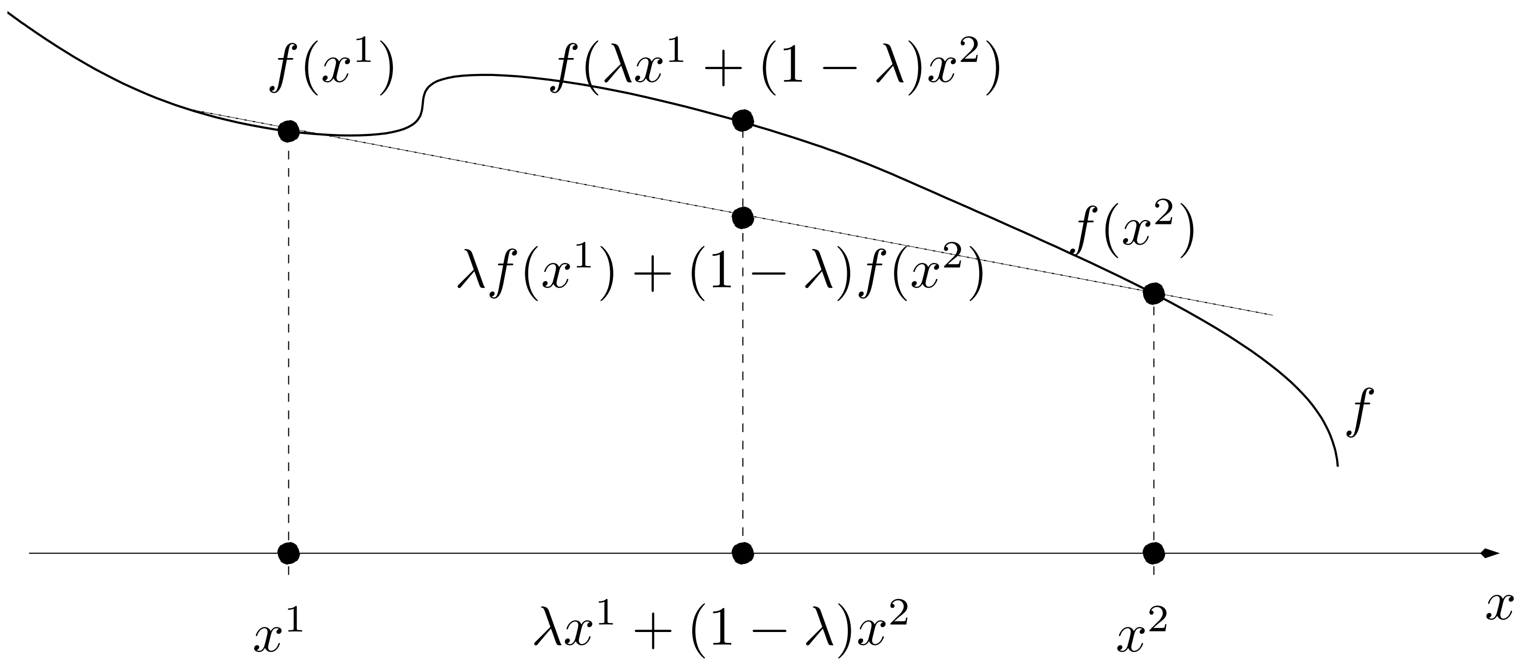

Figure 11: A non-convex function.

- Sums of convex functions are convex.
- Composite function: $\boldsymbol{x} \mapsto f(g(\boldsymbol{x}))$
- *Suppose that* $S \subseteq \mathbb{R}^n$ *and* $P \subseteq \mathbb{R}$. *Let further* $g : S \to \mathbb{R}$ *be a function which is convex on* S, *and* $f : P \to \mathbb{R}$ *be convex and non-decreasing* ($y \geq x \Longrightarrow f(y) \geq f(x)$) *on* P. *Then, the composite function* $f(g)$ *is convex on the set* $\{ \boldsymbol{x} \in \mathbb{R}^n \mid g(\boldsymbol{x}) \in P \}$.
- The function $\boldsymbol{x} \mapsto -\log(-g(\boldsymbol{x}))$ is convex on the set $\{ \boldsymbol{x} \in \mathbb{R}^n \mid g(\boldsymbol{x}) < 0 \}$.

Epigraphs

- Characterize convexity of a function on $\mathbb{R}^n$ by the convexity of its *epigraph* in $\mathbb{R}^{n+1}$. [*Note:* the *graph* of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the boundary of epi f.]

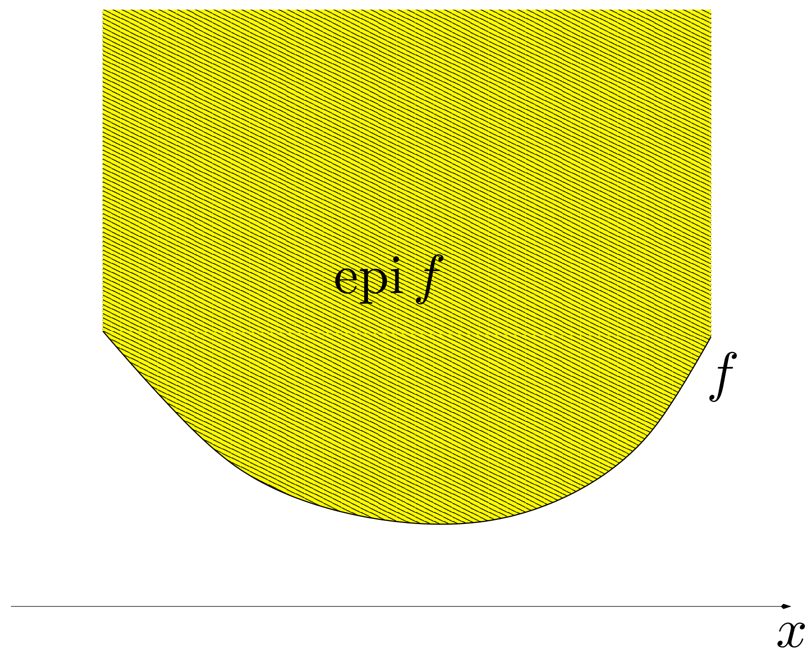

Figure 12: A convex function and its epigraph.

- The epigraph of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is the set

$$\operatorname{epi} f := \{ (\boldsymbol{x}, \alpha) \in \mathbb{R}^{n+1} \mid f(\boldsymbol{x}) \leq \alpha \}.$$

  The epigraph of the function f restricted to the set $S \subseteq \mathbb{R}^n$ is

$$\operatorname{epi}_S f := \{ (\boldsymbol{x}, \alpha) \in S \times \mathbb{R} \mid f(\boldsymbol{x}) \leq \alpha \}.$$

- Connection between convex sets and functions; in fact the definition of a convex function stems from that of a convex set!

- *Suppose that* $S \subseteq \mathbb{R}^n$ *is a convex set. Then, the function* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *is convex on* S *if, and only if, its epigraph restricted to* S *is a convex set in* $\mathbb{R}^{n+1}$.

Convexity characterizations in C^1

- C^1: Differentiable once, gradient continuous.
- *Let $f \in C^1$ on an open convex set S.*
  (a) f *is convex on* $S \Longleftrightarrow f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \nabla f(\boldsymbol{x})^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x})$, $\boldsymbol{x}, \boldsymbol{y} \in S$.
  (b) f *is convex on* $S \Longleftrightarrow [\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})]^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{y}) \geq 0, \boldsymbol{x}, \boldsymbol{y} \in S$.
- (a): "Every tangent plane to the function surface lies on, or below, the epigraph of f", or, that "a first-order approximation is below f."
- (b) ∇f is "monotone on S." [*Note:* when $n = 1$, the result states that f is convex if and only if its derivative f' is non-decreasing, that is, that it is monotonically increasing.]
- Proofs use Taylor expansion, convexity and Mean-value Theorem.

- Figure 13 illustrates part (a).

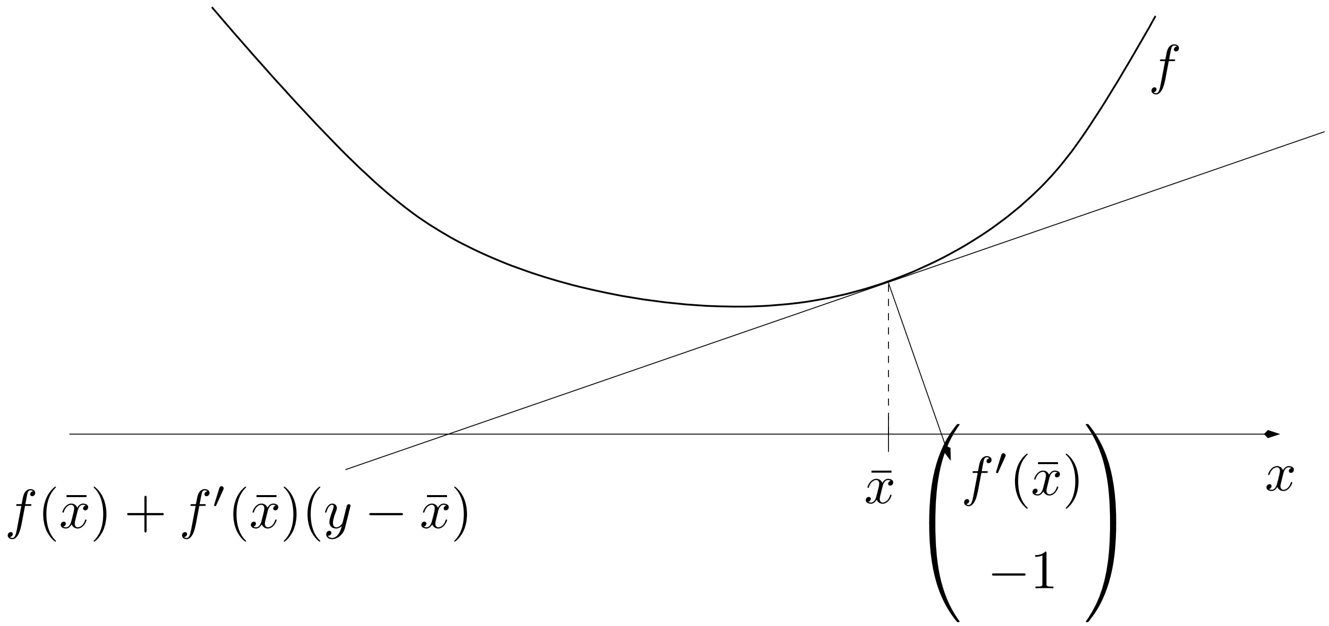

Figure 13: A tangent plane to the graph of a convex function.

Convexity characterizations in C^2

- *Let f be in C^2 on an open, convex set $S \subseteq \mathbb{R}^n$.*
  (a) *f is convex on $S \Longleftrightarrow \nabla^2 f(\boldsymbol{x})$ is positive semidefinite for all $\boldsymbol{x} \in S$.*
  (b) *$\nabla^2 f(\boldsymbol{x})$ is positive definite for all $x \in S \Longrightarrow f$ is strictly convex on S.*
- *Note:* $n = 1$, S is an open interval: (a) f is convex on S if and only if $f''(x) \geq 0$ for every $x \in S$; (b) f is strictly convex on S if $f''(x) > 0$ for every $x \in S$.
- Proofs use Taylor expansion, convexity and Mean-value Theorem.
- Not the direction $\Longleftarrow$ in (b)! $[f(x) = x^4$ at $x = 0]$
- Difficult to check convexity; matrix condition for every $\boldsymbol{x}$.
- Quadratic function: $f(\boldsymbol{x}) = (1/2)\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x} - \boldsymbol{q}^{\mathrm{T}}\boldsymbol{x}$ convex on $\mathbb{R}^n$ iff $\boldsymbol{Q}$ is psd ($\boldsymbol{Q}$ is the Hessian of f, and is independent of $\boldsymbol{x}$).

Convexity of feasible sets

- Let $g : \mathbb{R}^n \to \mathbb{R}$ be a function. The *level set* of g with respect to the value $b \in \mathbb{R}$ is the set

$$\text{lev}_g(b) := \{\, x \in \mathbb{R}^n \mid g(x) \leq b \,\}.$$

- Figure 14 illustrates a level set of a convex function.

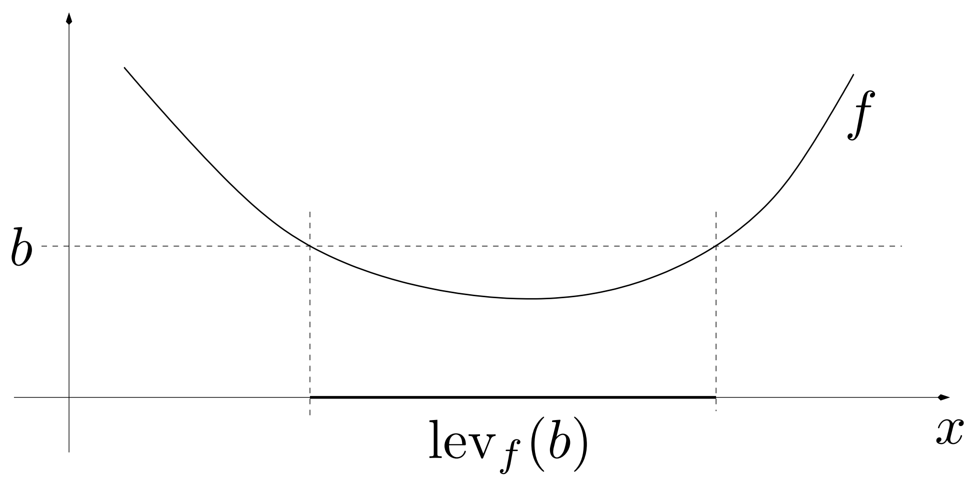

Figure 14: A level set of a convex function.

- *Suppose that the function $g : \mathbb{R}^n \to \mathbb{R}$ is convex. Then, for every value of $b \in \mathbb{R}$, the level set $\mathrm{lev}_g(b)$ is a convex set. It is moreover closed.*

  Proof.

- We speak of a *convex problem* when f is convex (minimization) and for constraints $g_i(\boldsymbol{x}) \leq 0$, the functions g_i are convex; and for constraints $h_j(\boldsymbol{x}) = 0$, the functions h_j are affine.

Euclidean projection

- The Euclidean projection of $\boldsymbol{w} \in \mathbb{R}^n$ is the nearest (in Euclidean norm) vector in S to $\boldsymbol{w}$. The vector $\boldsymbol{w} - \mathrm{Proj}_S(\boldsymbol{w})$ is *normal* to S.

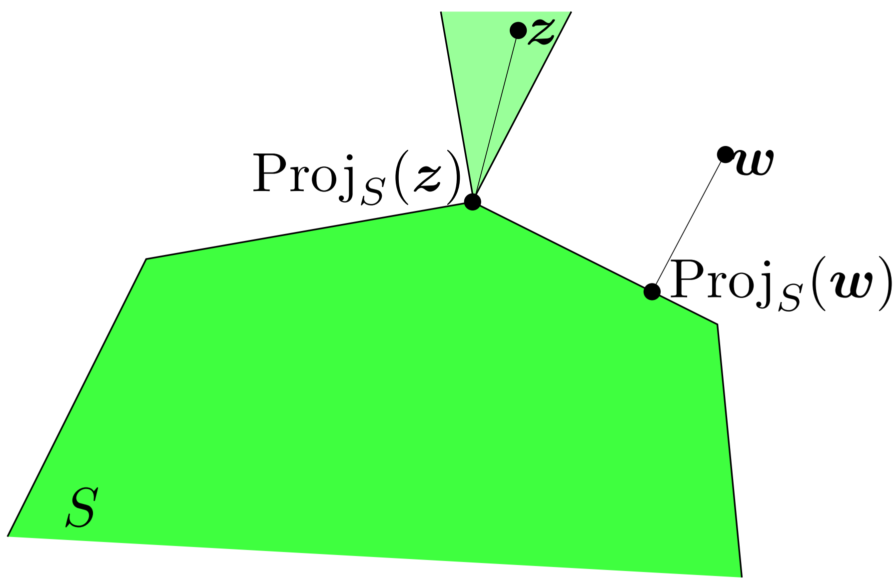

Figure 15: The projection of two vectors onto a convex set.

The distance function below is convex:

$$\mathrm{dist}_S(\boldsymbol{x}) := \|\boldsymbol{x} - \mathrm{Proj}_S(\boldsymbol{x})\|, \qquad \boldsymbol{x} \in \mathbb{R}^n.$$

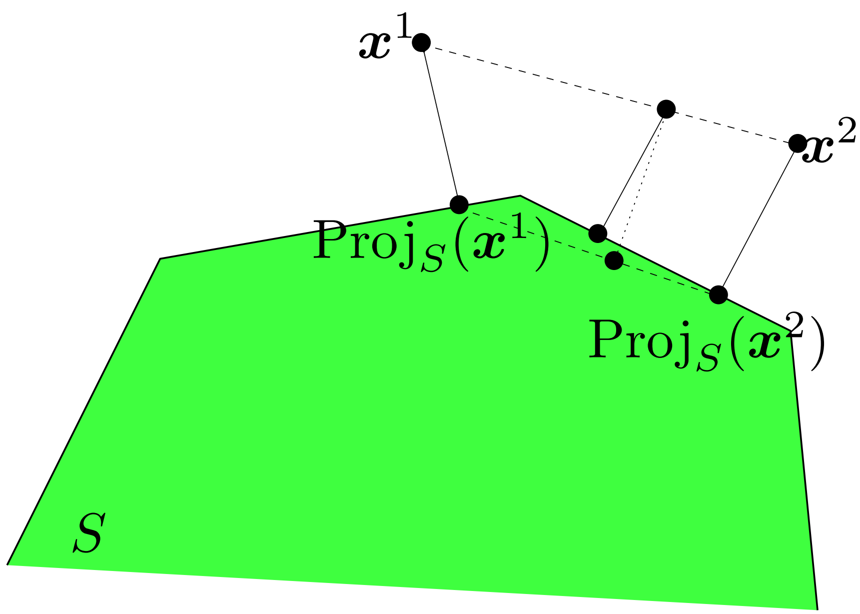

Figure 16: From the intermediate vector $\lambda\boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^2$ shown the distance to the vector $\lambda\mathrm{Proj}_S(\boldsymbol{x}^1) + (1-\lambda)\mathrm{Proj}_S(\boldsymbol{x}^2))$ [dotted line segment] clearly is longer than to its projection on S [solid line].