

Lecture 8: The Simplex method

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An algebraic derivation of the pricing step

$$\begin{aligned} z^* &= \text{infimum } \mathbf{c}^T \mathbf{x} &= \text{infimum } \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N \\ &\text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b}, &\text{subject to } \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}, \\ &\mathbf{x} \geq \mathbf{0}^n &\mathbf{x}_B \geq \mathbf{0}^m; \mathbf{x}_N \geq \mathbf{0}^{n-m} \\ &= \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b} + \text{infimum } [\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N}] \mathbf{x}_N \\ &\text{subject to } \mathbf{B}^{-1} \mathbf{b} - \mathbf{B}^{-1} \mathbf{N} \mathbf{x}_N \geq \mathbf{0}^m, \\ &\mathbf{x}_N \geq \mathbf{0}^{n-m} \end{aligned}$$

- $\mathbf{x}_N = \mathbf{0}^{n-m}$ feasible. Let $\tilde{\mathbf{c}}_N^T := \mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{N}$
- If *reduced cost* $\tilde{\mathbf{c}}_N \geq \mathbf{0}^{n-m}$ then $\mathbf{x}_N = \mathbf{0}^{n-m}$ is optimal

- If $\tilde{\mathbf{c}}_N \not\geq \mathbf{0}^{n-m}$ then $\exists j \in N$ with $\tilde{c}_j < 0$. Then the current point $\mathbf{x}_N = \mathbf{0}^{n-m}$ may be non-optimal
- Generate a feasible descent direction
- Choose one that leads to a neighboring extreme point
- Swap one variable in B for one in N
- Increase one variable in N from zero
- Choose j^* to be among $\arg \min_{j \in N} \tilde{c}_j$
- We have then decided on the search direction

The basis change

- What is this direction?
- In \mathbf{x}_N -space: $\mathbf{p}_N = \mathbf{e}_{j^*}$ (unit vector)
- In \mathbf{x}_B -space: $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_N \implies \mathbf{p}_B = -\mathbf{B}^{-1}\mathbf{N}\mathbf{p}_N = -\mathbf{B}^{-1}\mathbf{N}_{j^*}$
- So, search direction in \mathbb{R}^n :

$$\mathbf{p} = \begin{pmatrix} \mathbf{p}_B \\ \mathbf{p}_N \end{pmatrix} = \begin{pmatrix} -\mathbf{B}^{-1}\mathbf{N}_{j^*} \\ \mathbf{e}_{j^*} \end{pmatrix}$$

- Descent? Yes, because $\mathbf{c}^T \mathbf{p} = \tilde{c}_{j^*} < 0!$

- Feasible? Must check that $\mathbf{A}\mathbf{p} = \mathbf{0}^m$ and that $p_i \geq 0$ if $x_i = 0, i \in B$. The first true by construction:
- (a) $\mathbf{A}\mathbf{p} = \mathbf{B}\mathbf{p}_B + \mathbf{N}\mathbf{p}_N = -\mathbf{B}\mathbf{B}^{-1}\mathbf{N}_{j^*} + \mathbf{N}\mathbf{e}_{j^*} = \mathbf{0}^m$
- (b) Suppose that $\mathbf{x}_B > \mathbf{0}^m$. Then, at least a small step in \mathbf{p} keeps $\mathbf{x}_B \geq \mathbf{0}^m$.
But if there is an i^* with $(\mathbf{x}_B)_{i^*} = 0$ and $(\mathbf{p}_B)_{i^*} < 0$ then it is not a feasible direction
- Must then perform a basis change without moving! A *degenerate basis change*: swap x_{j^*} for x_{i^*} in the basis
- Otherwise (and normally), we utilize the unit direction

- Line search? Linear objective; move the maximum step!
- Maximum step: If $\mathbf{p}_B \geq \mathbf{0}^m$ there is no finite maximum step! We have found an extreme direction \mathbf{p} along which $\mathbf{c}^T\mathbf{x}$ tends to $-\infty$! *Unbounded solution*
- Otherwise: Some basic variables will reach zero eventually. Choose as the outgoing variable a variable $i \in B$ with minimum in

$$x_{j^*} := \underset{i \in B}{\text{minimum}} \left\{ \frac{(\mathbf{B}^{-1}\mathbf{b})_i}{(\mathbf{B}^{-1}\mathbf{N}_{j^*})_i} \mid (\mathbf{B}^{-1}\mathbf{N}_{j^*})_i > 0 \right\}$$

- Done. In the basis, replace i^* by j^* ; goto pricing step
- If $x_{j^*} = 0$ then the above corresponds to a “degenerate basis change”

Computational notes—how do we do all of this?

- Given basis matrix \mathbf{B} , solve

$$\mathbf{B}\mathbf{x}_B = \mathbf{b}$$

- Gives us BFS: $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$
- Pricing step: (a) Solve

$$\mathbf{B}^T\mathbf{y} = \mathbf{c}_B \implies \mathbf{y}^T = \mathbf{c}_B^T\mathbf{B}^{-1}$$

- (b) Calculate $\tilde{\mathbf{c}}_N^T = \mathbf{c}_N^T - \mathbf{y}^T\mathbf{N}$, the reduced cost vector
- Choose the incoming variable, x_{j^*}

- Outgoing variable: Solve

$$\mathbf{B}\mathbf{p}_B = -\mathbf{N}_{j^*}$$

- Quotient rule for $(\mathbf{B}^{-1}\mathbf{b})_i / (-\mathbf{p}_B)_i$ gives outgoing variable, x_{1^*} , and value of the new basic variable, x_{j^*}
- Note: Three similar linear systems in \mathbf{B} ! LU factorization + three triangular substitutions
- Factorizations can be updated after basis change rather than done from scratch
- LP solvers like Cplex and XPRESS-MP have excellent numerical solvers for linear systems
- Linear systems the bulk of the work in solving an LP

Convergence

- *If all of the basic feasible solutions are non-degenerate, then the Simplex algorithm terminates after a finite number of iterations*
- *Proof: (Rough argument) Non-degeneracy implies that the step length is > 0 ; hence, we cannot return to an old BFS once we have left it. There are finitely many BFSs*

- Degeneracy: Can actually lead to cycling—the same sequence of BFSs is returned to indefinitely!
- Remedy: Change the incoming/outgoing criteria!
Bland's rule: Sort variables according to some index ordering. Take the first possible index in the list.
Incoming variable first in the list with the right sign of the reduced cost; outgoing variable the first in the list among the minima in the quotient rule

Initial BFS: Phase I of the Simplex method

- If a starting BFS cannot be found, do the following:
- Suppose $\mathbf{b} \geq \mathbf{0}^m$. Introduce *artificial variables* a_i in every row (or rows without a unit column)
- Solve the following Phase-I problem:

$$\begin{aligned} \text{minimize } w &= (\mathbf{1}^m)^\top \mathbf{a} \\ \text{subject to } \mathbf{Ax} + \mathbf{I}^m \mathbf{a} &= \mathbf{b}, \\ \mathbf{x} &\geq \mathbf{0}^n, \\ \mathbf{a} &\geq \mathbf{0}^m \end{aligned}$$

- Possible cases: (a) $w^* = 0$, meaning that $\mathbf{a}^* = \mathbf{0}^m$ must hold. There is then a BFS in the *original* problem

- Start Phase-II, to solve the original problem, starting from this BFS
- (b) $w^* > 0$. The optimal basis then has some $a_i^* > 0$; due to the objective function construction, there exists no BFS in the original problem. The problem is then infeasible!
- What to do then? Modelling errors? Can be detected from the optimal solution. In fact, some LP problems are pure feasibility problems