EXERCISE 11: LINEARLY CONSTRAINTED NONLINEAR OPTIMIZATION

NICLAS ANDRÉASSON

The Frank-Wolfe algorithm

Step 0: Generate the starting point $\boldsymbol{x}_0 \in X$, for example by letting it be any extreme point in X. Set $k := 0$.

Step 1: Solve the problem to

$$\underset{y \in X}{\text{minimize}}\ z_k(\boldsymbol{y}) := \nabla f(\boldsymbol{x}_k)^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{x}_k). \tag{1}$$

Let $\boldsymbol{y}_k$ be a solution (extreme point) to this LP problem, and $\boldsymbol{p}_k := \boldsymbol{y}_k - \boldsymbol{x}_k$ be the search direction.

Step 2: Approximately solve the one-dimensional problem to minimize $f(\boldsymbol{x}_k + \alpha \boldsymbol{p}_k)$ over $\alpha \in [0, 1]$. Let α_k be the resulting step length.

Step 3: Let $\boldsymbol{x}_{k+1} := \boldsymbol{x}_k + \alpha_k \boldsymbol{p}_k$.

Step 4: If, for example, $z_k(\boldsymbol{y}_k)$ or α_k is close to zero, then terminate! Otherwise, let $k := k + 1$ and go to Step 1.

EXERCISE 1 (the Frank-Wolfe method). Consider the linearly constrainted nonlinear optimization problem to

$$\begin{aligned} \text{minimize} \quad & z = \frac{1}{2}(x_1 - \frac{1}{2})^2 + \frac{1}{2}x_2^2 \\ \text{subject to} \quad & x_1 \leq 1, \\ & x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

(a) Start at the extreme point $(1, 1)^{\mathrm{T}}$ and perform two iterations of the Frank-Wolfe algorithm to get the point $\boldsymbol{x}_2$!

(b) Is $\boldsymbol{x}_2$ optimal?

(c) Give upper and lower bounds for the optimal solution!

□

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The simplicial decomposition algorithm

Step 0: Generate the starting point $\boldsymbol{x}_0 \in X$, for example by letting it be any extreme point in X. Set $k := 0$. Let $\mathcal{P}_0 := \emptyset$.

Step 1: Let $\boldsymbol{y}^k$ be a solution (extreme point) to the LP problem (1). Let $\mathcal{P}_{k+1} := \mathcal{P}_k \cup \{k\}$.

Step 2: Let $\boldsymbol{\nu}_{k+1}$ be a solution to the *restricted master problem* to

$$\begin{aligned} &\text{minimize} \quad && f\left(\boldsymbol{x}_k + \sum_{i \in \mathcal{P}_{k+1}} \nu_i (\boldsymbol{y}^i - \boldsymbol{x}_k)\right) \\ &\text{subject to} \quad && \sum_{i \in \mathcal{P}_{k+1}} \nu_i \leq 1, \\ &&& \nu_i \geq 0, \qquad i \in \mathcal{P}_{k+1}. \end{aligned}$$

Step 3: Let $\boldsymbol{x}_{k+1} := \boldsymbol{x}_k + \sum_{i \in \mathcal{P}_{k+1}} (\boldsymbol{\nu}_{k+1})_i (\boldsymbol{y}^i - \boldsymbol{x}_k)$.

Step 4: If, for example, $z_k(\boldsymbol{y}^k)$ is close to zero, or if $\mathcal{P}_{k+1} = \mathcal{P}_k$, then terminate! Otherwise, let $k := k + 1$ and go to Step 1.

EXERCISE 2 (the simplicial decomposition method). Consider the linearly constrainted nonlinear optimization problem to

$$\begin{aligned} &\text{minimize} \quad && z = \frac{1}{2}(x_1 - \frac{1}{2})^2 + \frac{1}{2}x_2^2 \\ &\text{subject to} \quad && x_1 \leq 1, \\ &&& x_2 \leq 1, \\ &&& x_1, x_2 \geq 0. \end{aligned}$$

(a) Start at the extreme point $(1, 1)^{\mathrm{T}}$ and perform two iterations of the simplicial decomposition algorithm to get the point $\boldsymbol{x}_2$!

(b) Is $\boldsymbol{x}_2$ optimal?

□

EXERCISE 3 (finite convergence of the simplicial decomposition algorithm). Show that the simplicial decomposition algorithm converges in a finite number of steps! □

EXERCISE 4 (the gradient projection algorithm). Consider the problem to

$$\begin{aligned} &\text{minimize} \quad && f(\boldsymbol{x}) \\ &\text{subject to} \quad && \boldsymbol{x} \in X, \end{aligned}$$

where $X \subseteq \mathbb{R}^n$ is non-empty, closed and convex, and $f : \mathbb{R}^n \to \mathbb{R}$ is in C^1 on X. Let $\boldsymbol{x} \in X$, $\alpha > 0$, and

$$\boldsymbol{p} = \mathrm{Proj}_X[\boldsymbol{x} - \alpha \nabla f(\boldsymbol{x})] - \boldsymbol{x}.$$

Show that if $\boldsymbol{p} \neq \boldsymbol{0}^n$, then it defines a descent direction (this is exactly the direction used in the gradient projection algorithm)! □