Chalmers/GU
Mathematics

EXAM SOLUTION

TMA947/MAN280
APPLIED OPTIMIZATION

Date: 06–03–06
Examiner: Michael Patriksson

Question 1

(the Simplex method)

(2p) a) After adding two slack variables, a BFS cannot be found directly. We create the phase I problem through an added artificial variable a_1 in the second linear constraint; the value of a_1 is to be minimized. We use the BFS based on the variable pair (s_1, a_1) as the starting BFS for the phase I problem. One iteration with the simplex methods gives the optimal basis $x_B = (s_1, x_3)^T$, which is a BFS for the original problem.

Starting phase II with this BFS, in the first iteration x_1 is the only variable with negative reduced cost and is picked as the incoming variable. The minimum ratio test shows that s_1 should leave the basis. In the next iteration, all reduced costs are > 0 and we conclude that we have found a unique optimal BFS $(x_1, x_3)^T = \frac{1}{5}(24, 13)^T$. The corresponding optimal objective value is $z^* = -\frac{11}{5}$.

(1p) b) Strong duality guarantees that if one of the primal or dual problem has an optimal solution then so does the other. Hence, the answer is yes.

(3p) Question 2

(The Karush–Kuhn–Tucker conditions)

See the proof of Theorem 5.25.

Question 3

(true or false claims in optimization)

For each of the following three claims, your task is to decide whether it is true or false. Motivate your answers.

(1p) a) False.

(1p) b) False.

(1p) c) Yes. The facts imply that the feasible set "expands" in a direction which we are both interested (dual variable positive) and able (non-degeneracy) to follow.

Question 4

(nonlinear programming)

(1p) a) The problem is convex when f is convex and each function h_j is affine.

(2p) b) One possibility is that no CQ is satisfied at $\boldsymbol{x}^*$.

It could also be the case that there exists a local minimum which is also a KKT point close to $\boldsymbol{x}^*$. (For numerical reasons the algorithm has terminated prematurely, and so $\boldsymbol{x}^*$ may only be near-optimal, which means that the KKT conditions cannot be satisfied exactly.)

Question 5

(modelling)

Introduce the variables

$$x_{ijk} = \begin{cases} 1 \text{ if number k is chosen for the entry on row i, column j} \\ 0 \text{ else} \end{cases}, \tag{1}$$

which is defined for $i, j, k = 1, \dots, 9$. We need the constraints

$$\sum_i x_{ijk} = 1, \quad \forall j, k \tag{2}$$

$$\sum_j x_{ijk} = 1, \quad \forall i, k \tag{3}$$

$$\sum_k x_{ijk} = 1, \quad \forall i, j \tag{4}$$

$$\sum_{i=3p-2}^{3p} \sum_{j=3p-2}^{3p} x_{ijk} = 1, \quad \forall k,\ p = 1, 2, 3 \tag{5}$$

$$x_{ijk} \in \{0, 1\}, \quad \forall i, j, k \tag{6}$$

Equation 2 makes sure that each column contains each number, equation 3 that each row contains each number, equation 5 that each submatrix contains each number and equation 4 that each entry has exactly one number assigned to it. Equation 6 is a logic equation, saying that either is a number picked or not.

The objective function is

$$\min z = n - \sum_{ijk \in N} x_{ijk}, \tag{7}$$

where $z^* = 0$ means that the Sudoku problem could be solved with all the pre-assignments kept.

Question 6

(definitions)

(1p) a) See Definition 3.33.

(1p) b) See Step 2 on page 229.

(1p) c) See Definition 3.11.

Question 7

(Lagrangian duality for equality constrained problems)

(1p) a) At $\bar{\boldsymbol{\lambda}} \in \mathbb{R}^\ell$, a subgradient $\bar{\boldsymbol{\gamma}}$ of the concave function q is such that

$$q(\boldsymbol{\lambda}) \leq q(\bar{\boldsymbol{\lambda}}) + \bar{\boldsymbol{\gamma}}^{\mathrm{T}}(\boldsymbol{\lambda} - \bar{\boldsymbol{\lambda}}), \qquad \boldsymbol{\lambda} \in \mathbb{R}^\ell.$$

The subdifferential $\partial q(\bar{\boldsymbol{\lambda}})$ to q at $\bar{\boldsymbol{\lambda}}$ is the convex hull of all the subgradients $\bar{\boldsymbol{\gamma}}$ of q at $\bar{\boldsymbol{\lambda}}$.

(1p) b) Let $\bar{\boldsymbol{\lambda}} \in \mathbb{R}^\ell$, and $\bar{\boldsymbol{x}} \in X(\bar{\boldsymbol{\lambda}})$. Then,

$$\begin{aligned} q(\boldsymbol{\lambda}) = & \operatorname*{infimum}_{y \in X} L(\boldsymbol{y}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) + \boldsymbol{\lambda}^{\mathrm{T}} \boldsymbol{h}(\boldsymbol{x}) \\ & = f(\boldsymbol{x}) + \bar{\boldsymbol{\lambda}}^{\mathrm{T}} \boldsymbol{h}(\boldsymbol{x}) + (\boldsymbol{\lambda} - \bar{\boldsymbol{\lambda}})^{\mathrm{T}} \boldsymbol{h}(\boldsymbol{x}) \leq q(\bar{\boldsymbol{\lambda}}) + (\boldsymbol{\lambda} - \bar{\boldsymbol{\lambda}})^{\mathrm{T}} \boldsymbol{h}(\boldsymbol{x}), \end{aligned}$$

which implies that $\boldsymbol{h}(\boldsymbol{x}) \in \partial q(\boldsymbol{\lambda})$.

(1p) c) Suppose that $\mathbf{0}^\ell \in \partial q(\boldsymbol{\lambda})$. From b) then follows that $q(\boldsymbol{\lambda}) \geq q(\bar{\boldsymbol{\lambda}})$ for all $\bar{\boldsymbol{\lambda}} \in \mathbb{R}^\ell$.
