Chalmers/GU
Mathematics

EXAM

TMA947/MAN280
APPLIED OPTIMIZATION

Date:	04–03–08
Time:	House V, morning
Aids:	Text memory-less calculator
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Anton Evgrafov (0740-459022)
Result announced:	04–03–22 Short answers are also given at the end of the exam on the notice board for optimization in the MD building.

Exam instructions

When you answer the questions

State your methodology carefully.
Use generally valid methods and theory.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the Simplex method)

Consider the linear program

$$\begin{array}{llrrrl} \text{minimize} & z = & x_1 & +3x_2 & +x_3 & \\ \text{subject to} & & 2x_1 & -5x_2 & +x_3 & \leq -5, \\ & & 2x_1 & -x_2 & +2x_3 & \leq 4, \\ & & x_1, & x_2, & x_3 & \geq 0. \end{array}$$

(2p) a) Solve the problem by using Phase I & II of the Simplex method.

Hint: Some matrix inverses that can be useful when solving the problem are:

$$\begin{pmatrix} 2 & -5 \\ 2 & -1 \end{pmatrix}^{-1} = \frac{1}{8}\begin{pmatrix} -1 & 5 \\ -2 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 5 & 0 \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 0 \\ 1 & 5 \end{pmatrix},$$
$$\begin{pmatrix} -1 & -1 \\ 2 & 0 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix}^{-1} = \frac{1}{9}\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}.$$

(1p) b) Is the solution obtained unique? *Motivate!*

Question 2

(optimality conditions)

Consider the optimization problem to

$$\begin{aligned} \text{minimize} \quad & f(x,y) := \frac{1}{2}(x-2)^2 + \frac{1}{2}(y-1)^2, \\ \text{subject to} \quad & x - y \geq 0, \\ & y \geq 0, \\ & y(x-y) = 0, \end{aligned} \tag{1}$$

where $x, y \in \mathbb{R}$.

(1p) a) Find *all* points of global and local minimum, as well as *all* KKT-points.

Hint: Draw the problem graphically!

Is this a convex problem?

(1p) b) Demonstrate that the linear independence constraint qualification (LICQ) is violated at *every feasible point* of the problem (1).

The problem (1) can be solved as follows. Based on the original problem (1), we can formulate *two* optimization problems, both of which are convex and have one linear constraint. Having solved the two problems, the solution to the problem (1) is the solution with the best objective value.

Show *which* two problems should be solved.

Hint: Use the graphics used in a)!

(1p) c) In part b) we devised a "procedure" for solving the problem (1) in which two problems are solved and their respective optimal solutions compared. Generalize this procedure to the more general optimization problem to

$$
\begin{aligned}
\text{minimize} \quad & g(\boldsymbol{x}), \\
\text{subject to} \quad & \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} \geq b_i, \quad i = 1, \dots, n, \\
& x_i \geq 0, \quad i = 1, \dots, n, \\
& x_i(\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} - b_i) = 0, \quad i = 1, \dots, n,
\end{aligned}
$$

where $\boldsymbol{x} = (x_1, \dots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, $\boldsymbol{a}_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, $i = 1, \dots, n$, and $g : \mathbb{R}^n \to \mathbb{R}$ is a convex differentiable function.

How many problems do we need to solve, and what are their forms?

(3p) Question 3

(modelling)

Figure 1 below describes a production process by which we can make three products, A, B, and C, from the raw materials D, E, and F.

The numbers at the top of the figure provide the maximum sales and unit revenues for the three products. The numbers at the bottom indicate the raw materials used and the unit costs of the raw materials. We assume that the supply of the raw materials is unlimited.

The network structure shows the processing requirements for the products. The nodes represent operations that the intermediate products must pass through. Each node is labeled with the corresponding operation number. The arcs (links) represent the inputs to the operations. Product A requires one unit from operation 1. Product B requires one unit from operation 2. Product C requires one unit from operation 3.

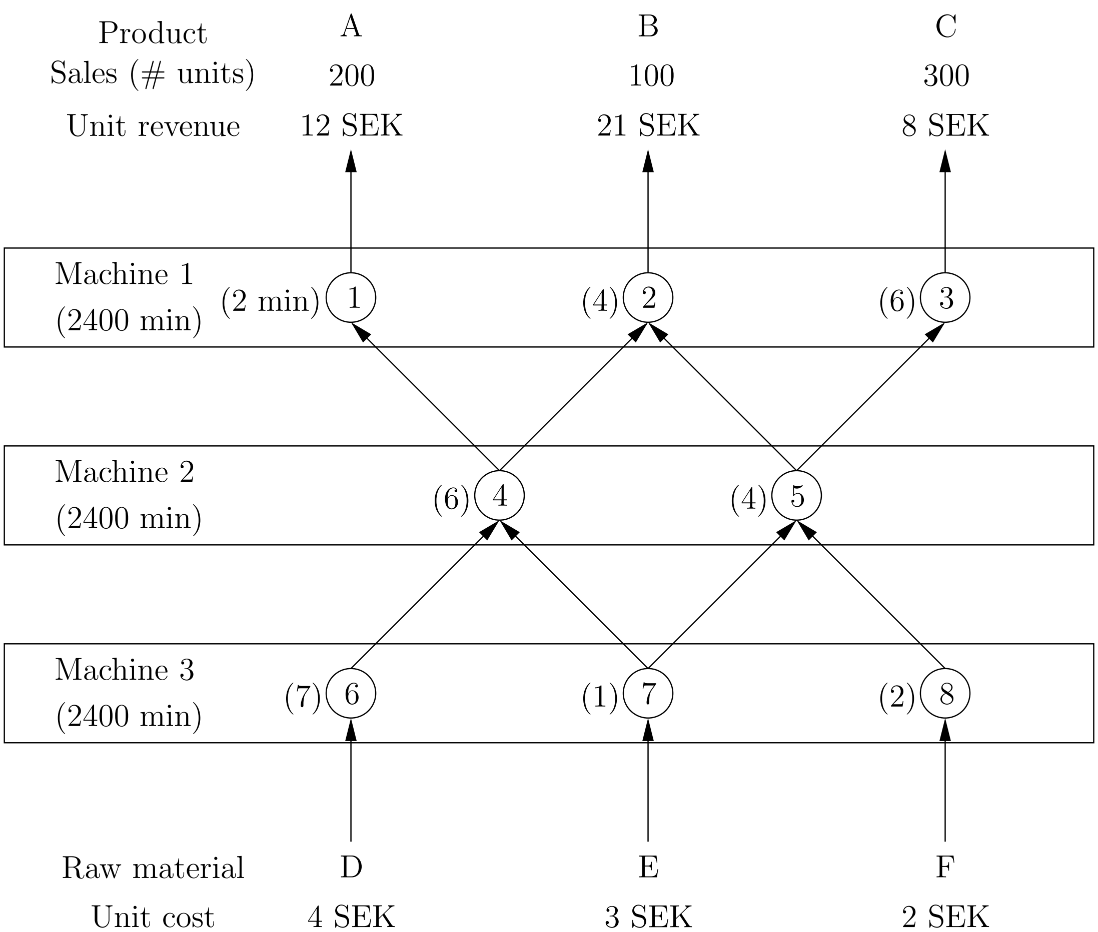

Figure 1: The production process.

The product at operation 1 is made from one unit passing out of operation 4. The product at operation 2 is made from one unit *each* of the products passing out of operations 4 and 5. The product at operation 3 is made from one unit passing out of operation 5.

A unit at operation 4 is made from one unit *each* from operations 6 and 7. A unit at operation 5 is made from one unit *each* from operations 7 and 8.

Finally, operation 6 requires one unit of raw material D. Operation 7 requires one unit of raw material E. Operation 8 requires one unit of raw material F.

The numbers adjacent to the nodes are the operation times in minutes. For example, operation 6 requires 7 minutes in order to produce one unit of output. The operations use time on Machine 1, Machine 2, and Machine 3. Each machine has a weekly capacity of 2400 minutes. The products share the capacities of the machines. For example, if 100 units of each product were produced, 1200 minutes would be used on machine 1. Note that one unit of product B needs 10 minutes of machine 2 because both operations 4 and 5 are needed for one unit of B.

Formulate a *linear integer programming model* (that is, if the integer requirements are relaxed we shall end up with an ordinary linear program) for finding the weekly production quantities that maximize the total income!

Hint: Introduce one variable for each arc (link) in the figure.

Question 4

(applications of the Newton algorithm)

(1p) a) Let a be a positive real number.

Consider the optimization problem to

$$\begin{aligned} &\text{minimize} \quad f(x) = ax - \log(x), \\ &\text{subject to} \quad x > 0. \end{aligned} \tag{1}$$

Prove that there exists a globally optimal solution to the problem (1), which is furthermore unique and equal to a^{-1}.

Motivate every step!

If you wish, you may replace the condition $x > 0$ with $x \geq 0$ in your analysis, as long as you are aware of the fact that $\log(0) = -\infty$.

(1p) b) Show that Newton's method with unit steps as applied to the problem (1) gives a computationally viable procedure for computing $x = a^{-1}$. That is, show that every iteration of Newton's method requires only additions (or subtractions) and multiplications to be performed; thus, we never need to perform divisions in order to compute the next iterate.

(*Note:* This idea is in fact used in the Intel Itanium processor!)

Construct an example (that is, choose some appropriate $a > 0$ and a starting point $x_0 > 0$) that satisfies the following requirements:

(i) Newton's method converges, that is, $\infty \neq \bar{x} = \lim_{k\to\infty} x_k$, but

(ii) $\bar{x} \neq a^{-1}$.

(*Note:* This illustrates the *local* nature of Newton's method, which is guaranteed to converge only when we start "near enough" to an optimal solution.)

(1p) c) Similarly to the previous parts, construct a convex optimization problem that can be used to calculate $x = a^{-1/2}$.

[That is, Newton's method with unit steps applied to your problem should not contain any other operations than additions (or subtractions) and multiplications.]

Question 5

(optimality conditions)

Farkas' Lemma can be stated as follows:

Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{b}$ an $m \times 1$ vector. Then exactly one of the systems

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b}, \\ \boldsymbol{x} &\geq \boldsymbol{0}^n, \end{aligned} \tag{I}$$

and

$$\begin{aligned} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} &\leq \boldsymbol{0}^n, \\ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} &> 0, \end{aligned} \tag{II}$$

has a feasible solution, and the other system is inconsistent.

(2p) a) Prove Farkas' Lemma.

(1p) b) Consider the problem to

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := \frac{1}{2}(x_1 - 1)^2 + \frac{1}{2}(x_2 - 1)^2, \\ \text{subject to} \quad & 2x_1 - x_2 = 0, \\ & 0 \leq x_1 \leq 2, \\ & 0 \leq x_2 \leq 2. \end{aligned}$$

Geometrically, it is not difficult to see that the vector $\bar{\boldsymbol{x}} := (0, 0)^{\mathrm{T}}$ cannot be an optimal solution to this problem. Your task is to prove this fact rigorously by using Farkas' Lemma, namely, prove that $\bar{\boldsymbol{x}} := (0, 0)^{\mathrm{T}}$ is not optimal to the above problem, by using Farkas' Lemma to show that there must exist a feasible descent direction with respect to f at $\bar{\boldsymbol{x}}$.

(3p) Question 6

(convexity)

Carathéodory's Theorem can be stated as follows:

Let $\boldsymbol{x} \in \operatorname{conv} V$, where $V \subseteq \mathbb{R}^n$. Then $\boldsymbol{x}$ can be expressed as a convex combination of $n+1$ or fewer points of V.

Prove Carathéodory's Theorem.

When you prove this result, you may make reference, without proof, to the following proposition:

Let $V \subseteq \mathbb{R}^n$. Then, $\operatorname{conv} V$ is the set of all convex combinations of points of V. ■

Question 7

(duality in linear and nonlinear optimization)

(1p) a) Consider the LP problem to

$$
\begin{array}{rrrl}
\text{minimize} & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} & +\boldsymbol{d}^{\mathrm{T}}\boldsymbol{v} & \\
\text{subject to} & \boldsymbol{A}_1\boldsymbol{x} & +\boldsymbol{B}\boldsymbol{v} & \geq \boldsymbol{b}_1, \\
& \boldsymbol{A}_2\boldsymbol{x} & & = \boldsymbol{b}_2, \\
& & \displaystyle\sum_{k=1}^{\ell} v_k & = a, \\
& \boldsymbol{x} & & \geq \boldsymbol{0}^n, \\
& & \boldsymbol{v} & \geq \boldsymbol{0}^\ell,
\end{array}
$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{v} \in \mathbb{R}^\ell$, $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{d} \in \mathbb{R}^\ell$, $\boldsymbol{A}_1 \in \mathbb{R}^{m_1 \times n}$, $\boldsymbol{A}_2 \in \mathbb{R}^{m_2 \times n}$, $\boldsymbol{B} \in \mathbb{R}^{m_1 \times \ell}$, $\boldsymbol{b}_1 \in \mathbb{R}^{m_1}$, $\boldsymbol{b}_2 \in \mathbb{R}^{m_2}$, and $a \in \mathbb{R}$.

State its LP dual problem.

(2p) b) Consider the strictly convex quadratic optimization problem to

$$
\begin{aligned}
\text{minimize} \quad & f(\boldsymbol{x}) := 2x_1^2 + x_2^2 - 4x_1 - 6x_2, && \text{(1a)} \\
\text{subject to} \quad & -x_1 + 2x_2 \leq 4. && \text{(1b)}
\end{aligned}
$$

For this problem, do the following:

[1] Explicitly state its Lagrangian dual function q and its Lagrangian dual problem, associated with the Lagrangian relaxation of the constraint (1b);

[2] Solve this Lagrangian dual problem and provide the optimal Lagrange multiplier μ^*;

[3] Provide the globally optimal solution $\boldsymbol{x}^*$ to the problem (1);

[4] Prove that strong duality holds, that is, prove that $q(\mu^*) = f(\boldsymbol{x}^*)$ holds.

Good luck!