Chalmers/GU
Mathematics

EXAM

TMA947/MAN280
OPTIMIZATION, BASIC COURSE

Date:	07–08–30
Time:	House V, morning
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Peter Lindroth (0762-721860)
Result announced:	07–09–13 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the Simplex method)

Consider the following linear program:

$$\begin{array}{lrrrrl} \text{minimize} & z = & x_1 & -x_2 & +x_3, & \\ \text{subject to} & & x_1 & +2x_2 & -2x_3 & \leq 0, \\ & & -x_1 & & +x_3 & \leq -1, \\ & & x_1, & x_2, & x_3 & \geq 0. \end{array}$$

(2p) a) Solve this problem by using phase I and phase II of the simplex method.
[*Aid:* Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for producing basis inverses.]

(1p) b) Is the solution unique? Motivate!

(3p) **Question 2**

(strong duality in linear programming)

Consider the following standard form of a linear program:

$$\begin{aligned} \text{minimize} \ & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned}$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{c}, \boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{b} \in \mathbb{R}^m$. State and prove the Strong Duality Theorem in linear programming.

Question 3

(exterior penalty method)

Consider the following problem:

$$\begin{aligned} &\text{minimize} \quad f(x) := \frac{1}{2}(x_1)^2 - x_1x_2 + (x_2)^2, \\ &\text{subject to} \quad x_1 + x_2 - 1 = 0. \end{aligned}$$

(1p) a) By applying the KKT conditions to this problem, establish its (unique) exact primal–dual solution.

(1p) b) Apply the standard exterior quadratic penalty method for this problem, and show that the sequence of (explicitly stated) subproblem solutions converges to the unique primal solution.

(1p) c) From the theory of exterior penalty methods provide the corresponding sequence of estimates of the Lagrange multiplier, and show that it converges to the solution provided in a).

Question 4

(true or false claims in optimization)

For each of the following three claims, your task is to decide whether it is true or false. Motivate your answers!

(1p) a) The vector $\boldsymbol{p} := (-1, 1)^{\mathrm{T}}$ is a descent direction for $f(\boldsymbol{x}) := (x_1 + x_2^2)^2$ at $\boldsymbol{x} = (1, 0)^{\mathrm{T}}$.

(1p) b) Suppose $f \in C^2$. If, at some iteration point $\boldsymbol{x} \in \mathbb{R}^n$ there exists a solution $\boldsymbol{p}$ to the search direction-finding problem of Newton's method then it defines a descent direction for f at $\boldsymbol{x}$.

(1p) c) Suppose $f \in C^1$. If $\boldsymbol{p} \in \mathbb{R}^n$ is a descent direction for f at $\boldsymbol{x}$ then the Armijo step length rule will in a finite number of steps provide a vector $\bar{\boldsymbol{x}}$ with $f(\bar{\boldsymbol{x}}) < f(\boldsymbol{x})$.

(3p) Question 5

(least-squares minimization)

Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^{m \times 1}$. If $m > n$, the system $\boldsymbol{Ax} = \boldsymbol{b}$ is over-determined and has in general no exact solution. Systems of this type appear e.g. in applications when a linear function is to be fitted to experimental data. The linear least-squares solution to the system is the vector

$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} ||\boldsymbol{Ax} - \boldsymbol{b}||_2$$

If the rank of $\boldsymbol{A}$ is n, motivate using optimality conditions and derive the closed form of the least-squares solution

$$\boldsymbol{x}^* = (\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{b}.$$

(3p) Question 6

(modelling)

You are assigned to model the planning of personnel for a construction project. The project will take 24 months to complete. To meet your requirements, you must have d_t construction workers working during month t, $t = 1, \cdots, 24$. If you recruit any workers in month t, you need to send them to a short introduction course. The course costs a fixed amount of k SEK regardless of the number of workers participating in the course, and w SEK for each participant. The salary for a construction worker is r SEK per month. Also, you cannot hire anyone for a period shorter than 3 months. Before the project starts ($t = 0$), you have no working personnel.

Your task is to minimize the total cost, subject to the above mentioned requirements. Formulate this as an optimization problem. [*Hint:* Let x_{ij} be the number of workers recruited at the beginning of month i to the end of month j.]

Question 7

(duality in linear and nonlinear optimization)

(1p) a) Consider the LP problem to

$$\text{minimize} \quad z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \quad + \boldsymbol{d}^{\mathrm{T}}\boldsymbol{v}$$

$$
\begin{aligned}
\text{subject to} \quad & \boldsymbol{A}_1\boldsymbol{x} + \boldsymbol{B}\boldsymbol{v} \geq \boldsymbol{b}_1, \\
& \boldsymbol{A}_2\boldsymbol{x} = \boldsymbol{b}_2, \\
& \sum_{k=1}^{\ell} v_k = a, \\
& \boldsymbol{x} \geq \boldsymbol{0}^n, \\
& \boldsymbol{v} \geq \boldsymbol{0}^\ell,
\end{aligned}
$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{v} \in \mathbb{R}^\ell$, $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{d} \in \mathbb{R}^\ell$, $\boldsymbol{A}_1 \in \mathbb{R}^{m_1 \times n}$, $\boldsymbol{A}_2 \in \mathbb{R}^{m_2 \times n}$, $\boldsymbol{B} \in \mathbb{R}^{m_1 \times \ell}$, $\boldsymbol{b}_1 \in \mathbb{R}^{m_1}$, $\boldsymbol{b}_2 \in \mathbb{R}^{m_2}$, and $a \in \mathbb{R}$.

State its LP dual problem.

(2p) b) Consider the strictly convex quadratic optimization problem to

$$
\begin{aligned}
\text{minimize} \quad & f(\boldsymbol{x}) := 2x_1^2 + x_2^2 - 4x_1 - 6x_2, && \text{(1a)} \\
\text{subject to} \quad & -x_1 + 2x_2 \leq 4. && \text{(1b)}
\end{aligned}
$$

For this problem, do the following:

[1] explicitly state its Lagrangian dual function q and its Lagrangian dual problem, associated with the Lagrangian relaxation of the constraint (1b);

[2] solve this Lagrangian dual problem and provide the optimal Lagrange multiplier μ^*;

[3] provide the globally optimal solution $\boldsymbol{x}^*$ to the problem (1); and

[4] prove that strong duality holds, that is, prove that $q(\mu^*) = f(\boldsymbol{x}^*)$ holds.

Good luck!