Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MAN280
OPTIMIZATION, BASIC COURSE

Date:	08–08–28
Time:	House M, morning
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Christoffer Cromvik (0762-721860)
Result announced:	08–04–02 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{array}{llrrrl} \text{minimize} & z = & x_1 + & x_2 & +3x_3, & \\ \text{subject to} & & & -\ x_2 & +3x_3 & \leq -1, \\ & & -2x_1 + & x_2 & -\ x_3 & \leq \ \ 1, \\ & & x_1, & x_2, & x_3 & \geq \ \ 0. \end{array}$$

(2p) a) Solve this problem by using phase I and phase II of the simplex method. [*Aid:* Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for producing basis inverses.]

(1p) b) Is the solution obtained unique? Motivate!

Question 2

(modelling)

Consider the following *portfolio selection problem*. An investor must choose a portfolio $\boldsymbol{x} = (x_1, x_2, \dots, x_n)^{\mathrm{T}}$, where x_j is the proportion of the assets allocated to the j:th security. The return on the portfolio has the mean value $\bar{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x}$ and the variance $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{V}\boldsymbol{x}$, where $\bar{\boldsymbol{c}}$ is the vector denoting mean returns and $\boldsymbol{V}$ is the matrix of covariances of the returns. The investor would (essentially) like to maximize his/her expected return, while at the same time minimize the variance and hence the risk.

A portfolio is called *efficient* (or *(weakly) Pareto optimal*) if there is no other portfolio having both a larger expected return and a smaller variance.

(2p) a) Formulate an optimization problem whose optimal solution corresponds to an efficient portfolio. (Different models are possible.) Motivate why your model leads to an efficient portfolio.

(1p) b) Suggest a systematic way of generating multiple efficient solutions.

Question 3

(interior penalty method)

Consider the following problem:

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := \tfrac{1}{2}(x_1+1)^2 + \tfrac{1}{2}(x_2+1)^2, \\ \text{subject to} \quad & x_1 \geq 0, \\ & x_2 \geq 0. \end{aligned}$$

(1p) a) Apply the logarithmic interior penalty method for this problem, and show that it converges to a limit point.

(1p) b) Is the limit point a KKT-point?

(1p) c) Under what conditions for a general nonlinear program can we be certain that the interior penalty method converges to a KKT-point?

Question 4

(necessary local and sufficient global optimality conditions)

Consider an optimization problem of the following general form:

$$\text{minimize } f(\boldsymbol{x}), \tag{1a}$$

$$\text{subject to } \boldsymbol{x} \in S, \tag{1b}$$

where $S \subseteq \mathbb{R}^n$ is nonempty, closed and convex, and $f : \mathbb{R}^n \to \mathbb{R}$ is in C^1 on S.

(1p) a) Establish the following result on the local optimality of a vector $\boldsymbol{x}^* \in S$ in this problem.

Proposition 1 (necessary optimality conditions, C^1 case) *If $\boldsymbol{x}^* \in S$ is a local minimum of f over S then*

$$\nabla f(\boldsymbol{x}^*)^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}^*) \geq 0, \qquad \boldsymbol{x} \in S \tag{2}$$

holds.

(2p) b) Establish the following result on the global optimality of a vector $\boldsymbol{x}^* \in S$ in this problem.

THEOREM 2 (necessary and sufficient global optimality conditions, C^1 case) *Suppose that* $f : \mathbb{R}^n \to \mathbb{R}$ *is convex on* S*. Then,*

$$\boldsymbol{x}^* \text{ is a global minimum of } f \text{ over } S \quad \Longleftrightarrow \quad (2) \text{ holds.}$$

Question 5

(Lagrangian duality)

Consider the following quadratic programming problem:

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := x_1^2 + 2x_2^2 - 4x_1 - 8x_2, \\ \text{subject to} \quad & x_1 + x_2 \leq 2, \qquad\qquad\qquad (1) \\ & x_1,\ x_2 \geq 0. \end{aligned}$$

We will attack this problem by using Lagrangian duality.

(1p) a) Consider Lagrangian relaxing the complicating constraint (1). Write down explicitly the resulting Lagrangian subproblem of minimizing the Lagrange function over the remaining constraints. Construct an explicit formula for the Lagrangian dual function. Establish that it is a concave function.

(1p) b) Solve the Lagrangian dual problem. State in particular the optimal solution, and the optimal value of the dual objective function.

(1p) c) Utilize the result in b) to generate an optimal solution to the original, primal, problem. Verify that strong duality holds. Is the primal optimal solution unique?

(3p) Question 6

(convexity)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \|\boldsymbol{A}\boldsymbol{x}\|_2^2 \leq 1, \end{aligned}$$

where $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, $\boldsymbol{A}$ is invertible and $\boldsymbol{c} \neq \boldsymbol{0}$. Show that the problem is convex and derive an explicit expression for the optimal solution.

Question 7

(linear programming duality and matrix games)

Let $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, and consider the canonical LP problem

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned}$$

and its associated dual LP problem. In the following, we denote the respective problem by (P) and (D).

(1p) a) If $m = n$ and $\boldsymbol{A}^{\mathrm{T}} = -\boldsymbol{A}$, we then say that the matrix $\boldsymbol{A}$ is *skew-symmetric.*

Suppose that in the problem (P), the matrix $\boldsymbol{A}$ is skew-symmetric and that $\boldsymbol{b} = -\boldsymbol{c}$ also holds. Establish that if an optimal solution to (P) exists, then $z^* = 0$ holds.

(2p) b) The problem studied in a) is known as a *self-dual* LP problem.

Consider again the canonical primal–dual pair (P), (D) of LP problems. Construct a self-dual LP problem in $n + m$ variables and $n + m$ linear constraints which is equivalent to (P), (D). By "equivalent" we refer to the property that any primal–dual optimal solutions $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ to the pair (P), (D) are obtained immediately as an optimal solution to the problem constructed. (In other words, we can solve any primal–dual pair of canonical LP problems as a self-dual LP problem in a higher dimension.)

Good luck!