Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MMG620
OPTIMIZATION, BASIC COURSE

Date:	09–08–27
Time:	House V, morning
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Jacob Sznajdman (0762-721860)
Result announced:	09–09–17 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{aligned}
\text{minimize} \quad & z = 2x_1 - x_2 + x_3, \\
\text{subject to} \quad & x_1 + 2x_2 - x_3 \leq 7, \\
& -2x_1 + x_2 - 3x_3 \leq -3, \\
& x_1, \quad x_2, \quad x_3 \geq 0.
\end{aligned}$$

(2p) a) Solve this problem by using phase I and phase II of the simplex method.

[*Aid:* Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for producing basis inverses.]

(1p) b) If the problem in a) is infeasible or unbounded, perform a small modification of the cost vector and/or the right-hand side vector such that the modified problem will have at least one optimal solution. If the problem has an optimal solution, state for which values of the first component (the one which is 7 now) of the right-hand side vector the optimal basis remains being the optimal one.

(3p) Question 2

(modeling)

A wind power company has n wind turbines located in an area. In order to perform maintenance, m maintenance units are located in different depots. Each day, the management obtains a list of maintenance activities to be performed at the different wind turbines. The data can be transformed into work hours that a crew has to spend at the site. Let d_i be the number of maintenance hours that must be spent on repairing turbine i. If the repairs will not be completed, the turbine can not run and will generate a loss of e_i SEK. Each crew can work a maximum number of 8 hours, but in order to perform repairs at a site, the crew has to travel to the site and return to the depot afterwards. Let c_{ij} be the time that crew j has to travel in order to reach turbine i. Since there only is a limited space in each turbine, a maximum of 2 crews may work on the same turbine during the day.

Create a linear mixed integer programming model (that is, a model which becomes a linear programming (LP) problem if any integrality requirements were to be removed) that schedules the maintenance work of the maintenance units at the turbines during one day so that the cost of the production losses are minimized.

(3p) ## Question 3

(optimality conditions)

Farkas' Lemma can be stated as follows:

Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{b}$ an $m \times 1$ vector. Then exactly one of the systems

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b}, \\ \boldsymbol{x} &\geq \boldsymbol{0}^n, \end{aligned} \tag{I}$$

and

$$\begin{aligned} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} &\leq \boldsymbol{0}^n, \\ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} &> 0, \end{aligned} \tag{II}$$

has a feasible solution, and the other system is inconsistent.

Prove Farkas' Lemma.

Question 4

(exterior penalty method)

Consider the following problem:

$$\begin{aligned} &\text{minimize} \quad f(x) := \frac{1}{2}(x_1)^2 - x_1 x_2 + (x_2)^2, \\ &\text{subject to} \quad x_1 + x_2 - 1 = 0. \end{aligned}$$

(1p) a) By applying the KKT conditions to this problem, establish its (unique) exact primal–dual solution.

(1p) b) Apply the standard exterior quadratic penalty method for this problem, and show that the sequence of (explicitly stated) subproblem solutions converges to the unique primal solution.

(1p) c) From the theory of exterior penalty methods provide the corresponding sequence of estimates of the Lagrange multiplier, and show that it converges to the dual solution provided in a).

Question 5

(topics in convexity)

Let there be given a function $f : \mathbb{R}^n \to \mathbb{R}$.

(2p) a) Let f be once continuously differentiable (that is, $f \in C^1$) on $\mathbb{R}^n$. Establish the following equivalence relation:

$$f \text{ is convex on } \mathbb{R}^n \iff f(y) \geq f(x) + \nabla f(x)^{\mathrm{T}}(y - x), \quad \text{for all } x, y \in \mathbb{R}^n.$$

(1p) b) Let f be twice continuously differentiable (that is, $f \in C^2$) on $\mathbb{R}^n$. Establish the following equivalence relation:

$$f \text{ is convex on } \mathbb{R}^n \iff \nabla^2 f(x) \text{ is positive semi-definite on } \mathbb{R}^n .$$

(3p) Question 6

(Lagrangian duality)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & -x_1 - 0.5x_2, \\ \text{subject to} \quad & (x_1 - 1)^2 + (x_2 - 1)^2 \geq 1, \\ & x_1^2 + x_2^2 \leq 1. \end{aligned}$$

Formulate the Lagrangian dual problem. Can we say something about convexity and differentiability of the dual problem? Let q be the Lagrangian dual function. Evaluate $q(1, 1/2)$ and $f(0, 1)$; what does this say about the optimal value of the primal problem f^*? Solve the primal problem graphically. Does this problem have a dual gap (i.e. is $f^* = q^*$) ? Motivate you answer!.

Question 7

(true or false claims in optimization)

For each of the following three claims, decide whether it is true or not. Motivate your answers! (Unless there is a clear motivation, no credits will be given.)

(1p) a) Consider the linear program

$$\begin{array}{lrl} \text{minimize} & z = & c_1x_1 + c_2x_2 + c_3x_3, \\ \text{subject to} & & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \le b_1, \\ & & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \le b_2, \\ & & a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \le b_3, \\ & & x_1, \quad x_2, \quad x_3 \ge 0, \end{array}$$

where $a_{ij} \in \mathbb{R},\ i = 1,2,3;\ j = 1,2,3$ and $b_j \in \mathbb{R},\ j = 1,2,3$ are such that there is at least one feasible point. Suppose that a fourth variable $x_4 \ge 0$ is added to the problem with cost coefficient c_4 and constraint coefficients $a_{j4},\ j = 1,2,3$.

<u>Claim:</u> No matter the values of c_4 and $a_{j4},\ j = 1,2,3$, the dual to the extended problem will never be unbounded.

(1p) b) <u>Claim:</u> The polyhedron in $\mathbb{R}^3$ defined by the following system

$$\begin{aligned} x_1 \quad\quad\quad\quad &\le 1, \\ 2x_2 \quad\quad &\le 2, \\ 2x_1 + 2x_2 + 2x_3 &\le 7, \\ x_3 &\le 1, \\ x_1 + x_2 + x_3 &\le 3, \end{aligned}$$

has an extreme point at $(1,1,1)^{\mathrm{T}}$.

(1p) c) Consider the non-linear program

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}), \\ \text{subject to} & \boldsymbol{g}(\boldsymbol{x}) \le \boldsymbol{0}^m. \end{array}$$

Suppose that there is a feasible point $\boldsymbol{x}^*$ fulfilling the KKT conditions, that a CQ is fulfilled at $\boldsymbol{x}^*$ and that there is a feasible point $\boldsymbol{y}$ arbitrarily close to $\boldsymbol{x}^*$ with $f(\boldsymbol{y}) > f(\boldsymbol{x}^*)$.

<u>Claim:</u> $\boldsymbol{x}^*$ is a local minimum to the problem.

Good luck!