EXERCISE 4: UNCONSTRAINED OPTIMIZATION

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EXERCISE 1 (descent directions). Consider the function

$$f(\boldsymbol{x}) = x_1^2 + x_1 x_2 - 4x_2^2 + 10.$$

Show that the direction $\boldsymbol{d} = (2, -1)^{\mathrm{T}}$ is not a descent direction at $\boldsymbol{x}^1 = (1, 1)^{\mathrm{T}}$. □

EXERCISE 2 (descent directions). Let $f : \mathbb{R}^n \to \mathbb{R}$ and $f \in C^2$. Assume that $\boldsymbol{x}^0$ is a point such that $\nabla f(\boldsymbol{x}^0) = \boldsymbol{0}^n$ and $\nabla^2 f(\boldsymbol{x}^0)$ is indefinite. Show that there exists a descent direction. □

EXERCISE 3 (Steepest descent method with exact line search). Consider the problem to

$$\underset{x \in \mathbb{R}^2}{\text{minimize}} \; f(\boldsymbol{x}) = (2x_1^2 - x_2)^2 + 3x_1^2 - x_2.$$

(a) Perform one iteration of the steepest descent method with exact line search (yielding the point $\boldsymbol{x}^1$). Start from the point $\boldsymbol{x}^0 = (1/2, 5/4)^{\mathrm{T}}$.
(b) Is the function convex in a small neighbourhood of $\boldsymbol{x}^1$?
(c) Will the method converge to a global optimum?

□

EXERCISE 4 (Newton's method with exact line search). Consider the problem to

$$\underset{x \in \mathbb{R}^2}{\text{minimize}} \; f(\boldsymbol{x}) = (x_1 + 2x_2 - 3)^2 + (x_1 - 2)^2.$$

(a) Start from $\boldsymbol{x}^0 = (0, 0)^{\mathrm{T}}$ and perform one iteration of Newton's method with exact line search (yielding $\boldsymbol{x}^1$).
(b) Are there any descent directions at the point $\boldsymbol{x}^1$?
(c) Is $\boldsymbol{x}^1$ an optimal solution?

□

EXERCISE 5 (Newton's method with Armijos's step length rule). Consider the problem to

$$\underset{x \in \mathbb{R}^2}{\text{minimize}} \; f(\boldsymbol{x}) = \frac{1}{2}(x_1 - 2x_2)^2 + x_1^4.$$

(a) Start from $\boldsymbol{x}^0 = (2, 1)^{\mathrm{T}}$ and use the fraction requirement $\mu = 0.3$. Perform one iteration of Newton's method with Armijo's step length rule.
(b) Determine the values of the fraction requirement $\mu \in (0, 1)$ for which Armijo's step length rule accept the step length $\alpha = 1$.

□

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EXERCISE 6 (Levenberg-Marquardt's method). Consider the problem to

$$\operatorname*{minimize}_{x\in\mathbb{R}^n} \; f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{q}^{\mathrm{T}}\boldsymbol{x},$$

where $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$ is symmetric and positive semi-definite, but not postivie definite. At iteration step k, by using Levenberg-Marquardt's method, we find the descent direction $\boldsymbol{p}_k$. Let

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{p}_k.$$

Show that $\boldsymbol{x}_{k+1}$ is an optimal solution to

$$\operatorname*{minimize}_{y\in\mathbb{R}^n} \; g(\boldsymbol{y}) = f(\boldsymbol{y}) + \frac{\gamma}{2}\|\boldsymbol{y} - \boldsymbol{x}_k\|^2,$$

where $\gamma > 0$ is the "shift". □