Chalmers/GU
Mathematics

**EXAM SOLUTION**

# TMA947/MAN280
# APPLIED OPTIMIZATION

**Date:** 05–08–25
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## Question 1

(the Simplex method)

**(2p)** a) After adding two slack variables, a BFS cannot be found directly. We create the phase I problem through an added artificial variable $a_1$ in the second linear constraint; the value of $a_1$ is to be minimized. We use the BFS based on the variable pair $(s_1, a_1)$ as the starting BFS for the phase I problem, terminating the simplex method with the optimal BFS given by $(s_1, x_2) = (3/2, 2)$, which is a BFS for the original problem.

Starting phase II with this BFS, the optimal basis for the problem is given by $(x_1, x_2) = (3/2, 1)$.

**(1p)** b) In the new problem the reduced cost vector for the non-basic variables is given by $\bar{\boldsymbol{c}}_N^{\mathrm{T}} = (-7/3, 1/3)$, indicating that the BFS is not optimal in the new problem. After one iteration of the simplex method, the optimal BFS reached is given by $(s_1, x_2) = (3/2, 2)$; hence $\boldsymbol{x}^* = (0, 2)^{\mathrm{T}}$.

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## Question 2

(the Karush–Kuhn–Tucker conditions)

**(1p)** a) $\boldsymbol{x}^* = (1, 1)^{\mathrm{T}}$ is the only feasible point, hence guaranteed to be globally optimal in the problem.

**(2p)** b) Both constraints are active at $\boldsymbol{x}^*$; their respectively normals (writing them as "$\leq$" constraints) are $(2, 2)^{\mathrm{T}}$ and $(-2, -2)^{\mathrm{T}}$, respectively. They are not linearly independent, thus violating the LICQ; the problem also violates the Slater CQ, since no interior point exists. The vector $-\nabla f(x^*) = (-1, 0)^{\mathrm{T}}$ cannot be written as a nonnegative linear combination of the normals of the active constraints, so the KKT conditions are not satisfied.

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## Question 3

(The Frank–Wolfe method)

**(2p)** a) $f$ is in $C^1$ and strictly convex on $X$ and $X$ is closed, convex and bounded, hence the problem has a unique optimal solution. Moreover, the Frank–Wolfe method converges to this point from any starting point. (The unconstrained minimum is $\frac{1}{15}(-8, 32)$.)

Starting at $\boldsymbol{x}_0 = (0, 0)^{\mathrm{T}}$, the algorithm proceeds as follows: $f(\boldsymbol{x}_0) = 0$; $\nabla f(\boldsymbol{x}_0) = (0, -2)^{\mathrm{T}}$; $\boldsymbol{y}_0 = (0, 1)^{\mathrm{T}}$ (for example); the lower bound $z(\boldsymbol{y}_0) = f(\boldsymbol{x}_0) + \nabla f(\boldsymbol{x}_0)^{\mathrm{T}}(\boldsymbol{y}_0 - \boldsymbol{x}_0) = -2$; $\boldsymbol{p}_0 = \boldsymbol{y}_0 - \boldsymbol{x}_0 = (0, 1)^{\mathrm{T}}$; $f(\boldsymbol{x}_0 + \alpha\boldsymbol{p}_0) = \frac{1}{2}\alpha^2 - 2\alpha$, which yields the unique minimum $\alpha = 1$ over the interval $\alpha \in [0, 1]$; $\boldsymbol{x}_1 = \boldsymbol{y}_0 = (0, 1)^{\mathrm{T}}$; $f(\boldsymbol{x}_1) = -3/2$; $\nabla f(\boldsymbol{x}_1) = (1/4, -1)^{\mathrm{T}}$; $\boldsymbol{y}_1 = \boldsymbol{x}_1 = (0, 1)^{\mathrm{T}}$; $z(\boldsymbol{x}_1) = f(\boldsymbol{x}_1) = -3/2$. The lower and upper bounds are equal, hence $\boldsymbol{x}_1 = (0, 1)^{\mathrm{T}} = \boldsymbol{x}^*$, with the optimal value $f^* = -3/2$.

**(1p)** b) The number of extreme points of $X$ is 4; hence, the maximum number of iterations of the simplicial decomposition method needed is also 4.

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## Question 4

(convexity of functions)

**(1p)** a) See Theorem 3.40(a) in AEP05.

**(2p)** b) See Theorem 3.41(a) in AEP05.

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## (3p) Question 5

(linear programming duality and optimality)

**(1p)** a) $\boldsymbol{x}^* = (1, 2)^{\mathrm{T}}$; the set of optimal dual solutions is $\{\, \boldsymbol{y} \in \mathbb{R}^3 \mid \boldsymbol{y} = (-1 + t, -2 + t, -t)^{\mathrm{T}}, \quad t \in [0, 1] \,\}$.

**(2p)** b) The three primal BFSs $(x_1, x_2, s_1)^{\mathrm{T}}$, $(x_1, x_2, s_2)^{\mathrm{T}}$, and $(x_1, x_2, s_3)^{\mathrm{T}}$ correspond to the dual basic solutions $\boldsymbol{y} = (0, -1, -1)^{\mathrm{T}}$, $\boldsymbol{y} = (1, 0, -2)^{\mathrm{T}}$, and $\boldsymbol{y} = (-1, -2, 0)^{\mathrm{T}}$, out of which the second one is infeasible—recall that the dual variables are restricted to be non-positive! Hence, the primal BFSs $(x_1, x_2, s_1)^{\mathrm{T}}$ and $(x_1, x_2, s_3)^{\mathrm{T}}$ are optimal, but the BFS $(x_1, x_2, s_2)^{\mathrm{T}}$ is not.

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## (3p) Question 6

(fixed points)

**(1p)** a) $\boldsymbol{x}_5 \approx 7.29000059778047948527991447491 37 \cdot 10^{-7}$; convergence is very rapid. Alternative (2) does not converge for any starting value $\boldsymbol{x}_0$.

**(2p)** b) With $g(x) = \frac{x^2+b}{-a}$ we can either establish the contraction property [hence utilize Banach's Theorem 4.34(a) in AEP05] or the convergence criterion that states that

$$|g'(x)| \leq \alpha < 1 \text{ holds on } S$$

(which is Exercise 4.9 in AEP05). Utilizing the latter, we obtain the condition that $2|\frac{x}{a}| \leq \alpha < 1$ holds on $S$, that is, that the value of $a$ is "large enough" in comparison with $x$ on $S$.

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## Question 7

(linear programming duality and matrix games)

**(1p)** a) Under the given conditions we have that

$$\begin{aligned} z^* &= \text{minimum } \{ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}^n \} \\ &= \text{maximum } \{ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \mid \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}, \quad \boldsymbol{y} \geq \boldsymbol{0}^m \} \\ &= \text{maximum } \{ (-\boldsymbol{c})^{\mathrm{T}}\boldsymbol{y} \mid -\boldsymbol{A}\boldsymbol{y} \leq -\boldsymbol{b}, \quad \boldsymbol{y} \geq \boldsymbol{0}^n \} \\ &= \text{maximum } \{ (-\boldsymbol{c})^{\mathrm{T}}\boldsymbol{y} \mid \boldsymbol{A}\boldsymbol{y} \geq \boldsymbol{b}, \quad \boldsymbol{y} \geq \boldsymbol{0}^n \} \\ &= -z^*, \end{aligned}$$

which implies that $z^* = 0$.

**(2p)** b) The self-dual skew symmetric LP problem sought is

$$\begin{aligned} &\text{minimize } \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}, \\ &\text{subject to } \begin{pmatrix} \boldsymbol{0}^{m\times n} & -\boldsymbol{A}^{\mathrm{T}} \\ \boldsymbol{A} & \boldsymbol{0}^{n\times m} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \geq \begin{pmatrix} -\boldsymbol{c} \\ \boldsymbol{b} \end{pmatrix}, \\ &\qquad\qquad (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0}^n \times \boldsymbol{0}^m. \end{aligned}$$

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