Chalmers/Gothenburg University
Mathematical Sciences

EXAM SOLUTION

TMA947/MAN280
APPLIED OPTIMIZATION

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Question 1

(the simplex method)

(2p) a) To transform the problem to standard form, the sign on the first constraint must be changed. Then subtract a non-negative slack variable s_1 in the first constraint and add one non-negative slack variable s_2 in the second. A BFS cannot be found directly, hence begin with phase 1 with an artificial variable $a \geq 0$ added in the first constraint - the slack varible s_2 can be used as the other basic variable. The objective is to minimize $w = a$. Start with the BFS given by (a, s_2). In the first iteration of the simplex algorithm, x_2 is the only variable with a negative reduced cost (-1), and is therefore the only eligable incoming variable. The minimum ratio test shows that either a or s_2 can be removed from the basis. By choosing a as the outgoing variable, we can proceed to phase 2.

The reduced costs in the first iteration of the phase 2 problem are

$$\tilde{\boldsymbol{c}}^{\mathrm{T}}_{(x_1,x_3,s_1)} = (1, 6, 1) \geq \mathbf{0},$$

and thus the optimality condition is fulfilled for the current basis. We have $\boldsymbol{x}^*_B = (1, 0)^{\mathrm{T}}$, or in the original variables, $\boldsymbol{x}^* = (x_1, x_2, x_3)^* = (0, 1, 0)^{\mathrm{T}}$, with the optimal value $z^* = 1$.

(1p) b) At the obtained optimal solution all reduced cost are strictly greater than zero, hence the obtained optimal solution must be unique.

Question 2

(modelling)

(2p) a) One possibility is the following formulation:

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := \mu(-\bar{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x}) + (1-\mu)\boldsymbol{x}^{\mathrm{T}}\boldsymbol{V}\boldsymbol{x}, \\ \text{subject to} \quad & \sum_{i=1}^{n} x_i = 1, \\ & x_i \geq 0, \qquad\qquad i = 1, \dots, n, \end{aligned}$$

where $\mu \in [0, 1]$ is a parameter balancing the two objectives. To show that the obtained solution $\boldsymbol{x}^*$ is efficient assume that it is not. Then there is another solution $\boldsymbol{y}^*$ such that $\bar{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{y}^* < \bar{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x}^*$ and $\boldsymbol{y}^{*\mathrm{T}}\boldsymbol{V}\boldsymbol{y}^* < \boldsymbol{x}^{*\mathrm{T}}\boldsymbol{V}\boldsymbol{x}^*$. But then obviously $f(\boldsymbol{y}^*) < f(\boldsymbol{x}^*)$ which is a contradiction to $\boldsymbol{x}^*$ being optimal.

Another possible formulation is:

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := -\bar{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{x}^{\mathrm{T}}\boldsymbol{V}\boldsymbol{x} \leq b, \\ & \sum_{i=1}^{n} x_i = 1, \\ & x_i \geq 0, \qquad\qquad i = 1, \dots, n, \end{aligned}$$

where b is a parameter setting the maximum value of the variance objective. Also here, the efficiency can be shown through contradiction.

(1p) b) By varying the parameter values μ and b, respectively, different efficient solutions can be found.

(3p) Question 3

(interior penalty method)

The logarithmic penalty function is

$$P(\boldsymbol{x}; \nu) = \tfrac{1}{2}(x_1 + 1)^2 + \tfrac{1}{2}(x_2 + 1)^2 - \nu \log x_1 - \nu \log x_2.$$

It is convex, and the first order optimality conditions is

$$\nabla P(\boldsymbol{x};\nu) = \begin{pmatrix} x_1 + 1 - \frac{\nu}{x_1} \\ x_2 + 1 - \frac{\nu}{x_2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives a unique optimal solution

$$\boldsymbol{x}^*(\nu) = \frac{-1 + \sqrt{1+4\nu}}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

due to the requirement $x_1 > 0, x_2 > 0$. As $\nu \to \infty$, we get $\boldsymbol{x}^* = (0,0)^{\mathrm{T}}$. It converges to a KKT-point. According to Theorem 13.6, we can guarantee convergence to a KKT-point if LICQ holds.

Question 4

(necessary local and sufficient global optimality conditions)

(1p) a) See Proposition 4.23.

(2p) b) See Theorem 4.24.

Question 5

(Lagrangian duality)

(1p) a) From the stationarity conditions for the Lagrangian we get that

$$x_1(\mu) = \begin{cases} (4-\mu)/2, & 0 \le \mu \le 4, \\ 0, & 4 \le \mu, \end{cases}$$

respectively,

$$x_2(\mu) = \begin{cases} (8-\mu)/4, & 0 \le \mu \le 8, \\ 0, & 8 \le \mu. \end{cases}$$

We then get the following expression for the Lagrangian dual function, to be maximized over $\mu \ge 0$:

$$q(\mu) = \begin{cases} -(3/8)\mu^2 + 2\mu - 12, & 0 \le \mu \le 4, \\ -(1/8)\mu^2 - 8, & 4 \le \mu \le 8, \\ -2\mu, & \mu \ge 8. \end{cases}$$

The corresponding derivatives then are:

$$q'(\mu) = \begin{cases} -(3/4)\mu + 2, & 0 \le \mu \le 4, \\ -(1/4)\mu, & 4 \le \mu \le 8, \\ -2, & \mu \ge 8. \end{cases}$$

It is clear that q is concave and differentiable for every $\mu \ge 0$. It is in fact strictly concave.

(1p) b) Setting $q'(\mu) = 0$ as a first trial, we obtain that $q'(\mu) = 0$ for $\mu = 8/3$. Since the dual problem is convex this is the optimal dual solution: $\mu^* = 8/3$. The corresponding objective value is $q(\mu^*) = -9\frac{1}{3}$.

(1p) c) The Lagrangian optimal solution in $\boldsymbol{x}$ for $\mu = \mu^*$ is, from a), $\boldsymbol{x} = (2/3, 4/3)^{\mathrm{T}}$. This is feasible in the primal problem, and $f(\boldsymbol{x}) = q(\mu^*)$ so it is also optimal, by the weak duality theorem. According to duality theory for convex problems over polyhedral sets, all primal optimal solutions are generated from Lagrangian optimal solutions given an optimal dual vector. Since $\boldsymbol{x}(\mu^*)$ here is the unique vector $\boldsymbol{x}^* = (2/3, 4/3)^{\mathrm{T}}$ this must also be the unique optimal solution to the primal problem.

(3p) Question 6

(convexity)

The objective function is convex and the constraint

$$\|\boldsymbol{A}\boldsymbol{x}\|_2^2 = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} \le 1$$

is convex since it is quadratic and the Hessian $\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}$ is positive definite, since $\boldsymbol{A}$ is invertible. (It is positive semidefinite even if it is not invertible, and hence convex.) This means that the problem is convex. The optimal solution can be computed from the KKT-conditions since e.g. Slater CQ holds.

$$\begin{aligned} c + 2\lambda \boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} &= \boldsymbol{0} \\ \|\boldsymbol{A}\boldsymbol{x}\|_2^2 &\le 1 \\ \lambda\left(\|\boldsymbol{A}\boldsymbol{x}\|_2^2 - 1\right) &= 0 \end{aligned}$$

Since the objective function is linear with nonzero gradient, the optimal solution must be at the boundary of the constraint. The first condition gives

$$\boldsymbol{x} = -\frac{1}{2\lambda}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\right)^{-1}\boldsymbol{c},$$

and hence

$$1 = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} \Rightarrow \lambda = \sqrt{\tfrac{1}{4}\boldsymbol{c}^{\mathrm{T}}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\right)^{-1}\boldsymbol{c}} = \tfrac{1}{2}\|\boldsymbol{A}^{-\mathrm{T}}\boldsymbol{c}\|_2.$$

The optimal solution is

$$\boldsymbol{x}^* = -\frac{1}{\|\boldsymbol{A}^{-\mathrm{T}}\boldsymbol{c}\|_2}\left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A}\right)^{-1}\boldsymbol{c}.$$

Question 7

(linear programming duality and matrix games)

(1p) a) Under the given conditions we have that

$$\begin{aligned} z^* &= \text{minimum } \{\, \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}^n \,\} \\ &= \text{maximum } \{\, \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} \mid \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}, \quad \boldsymbol{y} \geq \boldsymbol{0}^m \,\} \\ &= \text{maximum } \{\, (-\boldsymbol{c})^{\mathrm{T}}\boldsymbol{y} \mid -\boldsymbol{A}\boldsymbol{y} \leq -\boldsymbol{b}, \quad \boldsymbol{y} \geq \boldsymbol{0}^n \,\} \\ &= \text{maximum } \{\, (-\boldsymbol{c})^{\mathrm{T}}\boldsymbol{y} \mid \boldsymbol{A}\boldsymbol{y} \geq \boldsymbol{b}, \quad \boldsymbol{y} \geq \boldsymbol{0}^n \,\} \\ &= -z^*, \end{aligned}$$

which implies that $z^* = 0$.

(2p) b) The self-dual skew symmetric LP problem sought is

$$\begin{aligned} &\text{minimize } \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} - \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y}, \\ &\text{subject to } \begin{pmatrix} \boldsymbol{0}^{m\times n} & -\boldsymbol{A}^{\mathrm{T}} \\ \boldsymbol{A} & \boldsymbol{0}^{n\times m} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \geq \begin{pmatrix} -\boldsymbol{c} \\ \boldsymbol{b} \end{pmatrix}, \\ &\qquad\qquad\qquad\qquad (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0}^n \times \boldsymbol{0}^m. \end{aligned}$$
