Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MMG620
OPTIMIZATION, BASIC COURSE

Date:	09–04–14
Time:	House V, morning
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Adam Wojciechowski (0762-721860)
Result announced:	09–05–07
	Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{array}{lrcl} \text{minimize} & z = 4x_1 + 2x_2 + x_3, & & \\ \text{subject to} & 2x_1 \qquad + x_3 & \geq & 3, \\ & 2x_1 + 2x_2 + x_3 & = & 5, \\ & x_1, \quad x_2, \quad x_3 & \geq & 0. \end{array}$$

(2p) a) Solve this problem by using phase I and phase II of the simplex method.
[*Aid:* Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for producing basis inverses.]

(1p) b) Suppose the problem above describes the cost for a production plan, and that the constraints correspond to some sort of demands that have to be satisfied. Motivate (without re-solving the problem!) how much it would be worth to decrease each of the right-hand side components (one by one) with a small number $\varepsilon > 0$.

(3p) **Question 2**

(optimality conditions)

Farkas' Lemma can be stated as follows:

Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{b}$ an $m \times 1$ vector. Then exactly one of the systems

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b}, \\ \boldsymbol{x} &\geq \boldsymbol{0}^n, \end{aligned} \tag{I}$$

and

$$\begin{aligned} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} &\leq \boldsymbol{0}^n, \\ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} &> 0, \end{aligned} \tag{II}$$

has a feasible solution, and the other system is inconsistent.

Prove Farkas' Lemma.

(3p) Question 3

(the Frank–Wolfe algorithm)

As applied to the problem of minimizing a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ over a non-empty and bounded polyhedral set $X \subset \mathbb{R}^n$, the Frank–Wolfe method is defined, in short, thus: provide a first feasible solution $\boldsymbol{x}_0$ to the problem, and let $k := 0$; for given $\boldsymbol{x}_k$, solve the LP problem to minimize $\nabla f(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{y}$ over $\boldsymbol{y} \in X$, and let $\boldsymbol{y}_k$ be an optimal solution to this problem. If the value of $\nabla f(\boldsymbol{x}_k)^{\mathrm{T}}(\boldsymbol{y}_k - \boldsymbol{x}_k)$ is (near) zero, then terminate with $\boldsymbol{x}_k$ being a (near-)stationary point, otherwise let $\boldsymbol{p}_k := \boldsymbol{y}_k - \boldsymbol{x}_k$ and perform a line search in the value of f along the direction $\boldsymbol{p}_k$ from $\boldsymbol{x}_k$, with a maximum step length of 1. Let the resulting vector be $\boldsymbol{x}_{k+1} := \boldsymbol{x}_k + \alpha_k \boldsymbol{p}_k$, where α_k is the step length obtained in the line search. Let finally $k := k + 1$, and repeat.

Consider the following constrained nonlinear minimization problem:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(\boldsymbol{x}) := \frac{1}{2}(x_1^2 + x_2^2), \\ \text{subject to} \quad & \begin{cases} -1 \leq x_1 \leq 2, \\ 1 \leq x_2 \leq 1. \end{cases} \end{aligned}$$

Starting at the point $\boldsymbol{x}_0 = (2, 1)^{\mathrm{T}}$, perform one step of the Frank–Wolfe method. (Feel free to plot the problem and perform the algorithmic steps graphically, as long as you describe all the steps in detail with mathematical notation also.)

Is the point obtained an optimal solution? Why/why not? Explain in detail.

(3p) Question 4

(modeling)

A chocolate producer wants to plan the yearly production of chocolate. He has to fulfill the demand for chocolate in each month according to Table 1. In order to produce 1 kg of chocolate he needs 0.7 kg cocoa and 0.3 kg sugar. He has the possibility to sign a deal with an importer for monthly deliveries of sugar and cocoa; the importer will then deliver the same amount of sugar and cocoa each month (the amount is decided by the chocolate producer, but has to be equal for all months). He can also buy the goods for a higher price at the local market, but has then the possibility to buy different amounts each month. The prices are presented in Table 2. If there are goods left after a month's production, they can be stored until the next month. There is, however, a maximal storage capacity of 100 kg.

Introduce appropriate constants and variables, and create a Linear Programming model that minimizes the yearly production costs.

jan	feb	mar	apr	may	jun	jul	aug	sep	okt	nov	dec
300	230	270	500	150	170	140	230	300	270	350	700

Table 1: Demand of chocolate in kg for each month.

	import	market
cocoa	50	70
sugar	10	12

Table 2: Prices of goods in SEK/kg from import and local market.

(3p) Question 5

(gradient projection)

The gradient projection algorithm is a generalization of the steepest descent method to problems defined over convex sets. Given a point $\boldsymbol{x}_k$ the next point is obtained according to $\boldsymbol{x}_{k+1} = \text{Proj}_X[\boldsymbol{x}_k - \alpha \nabla f(\boldsymbol{x}_k)]$, where X is the convex set over which we minimize, α is the step length and $\text{Proj}_X(\boldsymbol{y}) = \arg\min_{\boldsymbol{x} \in X} \|\boldsymbol{x} - \boldsymbol{y}\|$ (i.e., the closest point to $\boldsymbol{y}$ in X). Note that if $X = \mathbb{R}$ then the method reduces to steepest descent.

Consider the optimization problem to

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) = (x_1 + x_2)^2 + 3(x_1 - x_2)^2, \\ \text{subject to} \quad & (x_1 - 1)^2 + (x_2 - 2)^2 \leq 1. \end{aligned}$$

Start at the point $x^0 = (1\ 2)^{\mathrm{T}}$ and perform two iterations of the gradient projection algorithm using step length $\alpha = 1/4$. Note that the special form of the feasible region X makes projection very easy! Is the point obtained a global/local optimum? Motivate why/why not!

(3p) Question 6

(a simple optimization problem)

In a recent optimization exam at a Swedish technical university, the following optimization problem was addressed:

$$\begin{aligned}
\text{maximize} \quad & f(\boldsymbol{x}) := \sum_{j=1}^{n} a_j/x_j, \\
\text{subject to} \quad & \sum_{j=1}^{n} \log x_j \leq b, \\
& \quad x_j > 0, \quad i = 1, \ldots, n,
\end{aligned}$$

where $a_j > 0$ for all j and $b > 0$.

The students were asked to derive the optimal solution to this problem through a Lagrangian relaxation of the first constraint, and by then solving the resulting dual problem. Explain what is wrong with that exam question. In other words, prove that there does not exist an optimal solution to this problem.

[*Hint:* Utilize the KKT conditions.]

(3p) Question 7

(polyhedral theory – LP duality)

Consider the polyhedron $P := \{ \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}^n \}$, where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$. Suppose that P is non-empty and that it is contained in some ball, i.e., $\exists M \in \mathbb{R}$ such that $P \subset \{ \boldsymbol{x} \in \mathbb{R}^n \mid ||\boldsymbol{x}|| \leq M \}$.

Show that the polyhedron $Q \subset \mathbb{R}^{n+m}$ defined as all points $\boldsymbol{z} \in \mathbb{R}^{n+m}$ fulfilling the inequalities below is non-empty for all vectors $\boldsymbol{c} \in \mathbb{R}^n$.

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{0}^{m\times m} \\ \boldsymbol{0}^{m\times n} & \boldsymbol{A}^{\mathrm{T}} \\ -\boldsymbol{I}^{n\times n} & \boldsymbol{0}^{n\times m} \\ \boldsymbol{0}^{m\times n} & \boldsymbol{I}^{m\times m} \\ \boldsymbol{c}^{\mathrm{T}} & -\boldsymbol{b}^{\mathrm{T}} \end{pmatrix} \boldsymbol{z} \leq \begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{c} \\ \boldsymbol{0}^n \\ \boldsymbol{0}^m \\ 0 \end{pmatrix}$$

Good luck!