Ex 0.1

a) $\nabla(x^T A x)$ give 2 min

$$\left(\nabla(x^T A x)\right)_k = \frac{d}{dx_k} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i A_{ij} x_j$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{d}{dx_k} x_i A_{ij} x_j = \frac{d}{dx_k} x_k A_{kk} x_k$$

$$\sum_{\substack{j=1 \\ j\neq k}}^{n} \frac{d x_k}{dx_k} A_{kj} x_j + \sum_{\substack{i=1 \\ i\neq k}}^{n} \frac{d x_i}{dx_k} A_{ik} x_k$$

$$+ \sum_{\substack{j=1 \\ j\neq k}}^{n} \sum_{\substack{i=1 \\ i\neq k}}^{n} \frac{d}{dx_k} x_i A_{ij} x_j$$

$$= 2 x_k A_{kk} + \sum_{\substack{j=1 \\ j\neq k}}^{n} A_{kj} x_j + \sum_{\substack{i=1 \\ i\neq k}} x_i A_{ik}$$

$$= \sum_{j=1}^{n} A_{kj} x_j + \sum_{i=1}^{n} x_i A_{ik}$$

$$(Ax)_k = \sum_{j=1}^{n} A_{kj} x_j \qquad (A^T x)_k = \sum_{i=1}^{n} A_{ik} x_i$$

$$\therefore \nabla(x^T A x) = Ax + A^T x$$

b) $\nabla^2 (x^T A x)$

$$\left(\nabla^2 (x^T A x)\right)_{\ell k} = \frac{d}{dx_\ell} \frac{d}{dx_k} x^T A x$$

$$= \frac{d}{dx_\ell} \left(\sum_{j=1}^{n} A_{kj} x_j + \sum_{i=1}^{n} x_i A_{ik} \right)$$

$$= A_{k\ell} + A_{\ell k}$$

$$\therefore \nabla^2 (x^T A x) = A + A^T$$

Ex 0.21

Def: $A \in \mathbb{R}^{n \times n}$ is pos (semi) definite

$\iff x^T A x > 0 \quad (x^T A x \geq 0) \quad \forall x \in \mathbb{R}^n$

Prop: A pos. def. $\iff \lambda_i > 0 \quad \forall i$
where λ_i are eigenvalues to A.

$\Rightarrow$ Let v_i eigenvector to A.

$$0 \underset{\substack{\text{def. } A \\ \text{pos. def.}}}{<} v_i^T A v_i \underset{\substack{v_i \\ \text{eigenv.}}}{=} \lambda_i v_i^T v_i = \lambda_i \underbrace{\|v_i\|^2}_{\substack{\vee \\ 0}}$$

$\Rightarrow \lambda_i > 0$

$\Leftarrow$ Let $V = \begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix}$ v_i ON-eigenvectors

spectral thm: $A = V D V^T$ $\quad D = \begin{bmatrix} \lambda_1 & 0 & - & 0 \\ 0 & \ddots & & | \\ | & & \ddots & 0 \\ 0 & - & 0 & \lambda_n \end{bmatrix}$

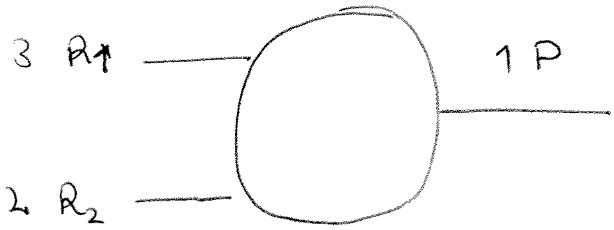
Let $x \in \mathbb{R}^n$, $y = V^T x$

$$x^T A x = x^T V D V^T x = y^T D y = \sum_i y_i^2 \lambda_i > 0$$

$\blacksquare$

(similar semi def. matr.)

Ex 1.1



variables: x_1 - purchase of R_1

x_2 - — " — R_2

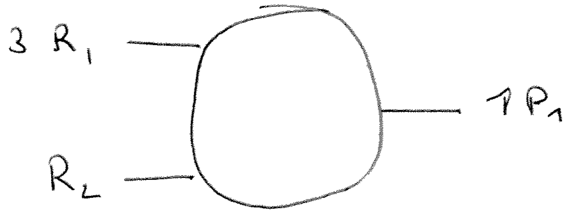
y - production of P

$$3y \leq x_1$$
$$2y \leq x_2$$

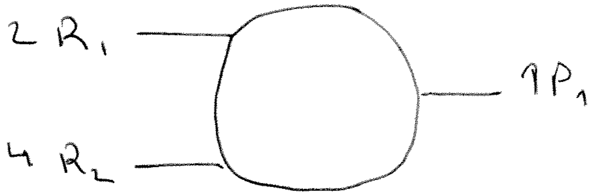
$$y, x_1, x_2 \geq 0$$

Ex 1.2

Process 1



Process 2



R_i costs r_i and has limited supply s_i

P_i is sold for price of p_i and a demand d_i exists.

$i = 1, 2$

Formulate an LP:

variables:

x_1 - purch. of R_1

x_2 - — " — R_2

y_1 - production of P_1

y_2 - production of P_2

$\boxed{\text{give 2 min}}$

objective function (cost − ~~gain~~ profit):

$$\min \; x_1 r_1 + x_2 r_2 - y_1 p_1 - y_2 p_2$$

constraints

$$0 \le x_1 \le s_1$$
$$0 \le x_2 \le s_2$$
$$0 \le y_1 \le d_1$$
$$0 \le y_2 \le d_2$$

$$3y_1 + 2y_2 \le x_1$$
$$y_1 + 4y_2 \le x_2$$

Ex 3. Consider the problem in Ex 2.
modification: before any production
is possible, a cost c_i must be paid.

New model:

$$\min \; p_1 y_1 + p_2 y_2 - x_1 r_1 - x_2 r_2 + c_1 z_1 + c_2 z_2$$

$\vdots$ same con.

$$y_1 \leq d_1 z_1$$
$$y_2 \leq d_2 z_2$$

$$z_1, z_2 \in \{0, 1\}$$

This is no longer an LP!
(it's a IP)

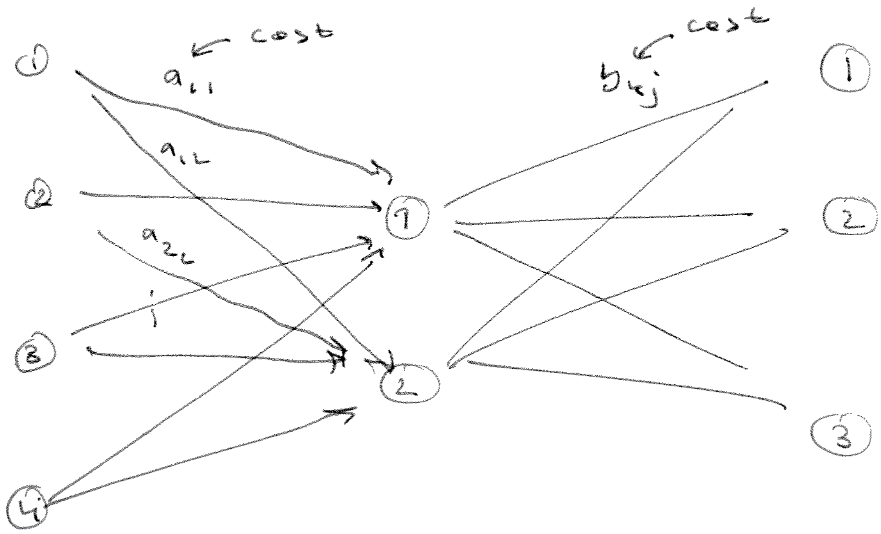
Ex 4, Transshipment problem

N sources

M demand centers

I intermediate nodes

$N = 4 \quad I = 2 \quad M = 3$



max cap. s_i demand d_j

formulate an LP that minimizes the costs.

variables $\quad x_{ik}$ = amount goods from i to k

$i = 1, \dots, 4 \qquad k = 1, \dots, 2$

y_{kj} = amount goods from k to j

$k = 1, 2 \qquad j = 1, 2, 3$

$$\min \sum_{i=1}^{4} \sum_{k=1}^{2} a_{ik} x_{ik} + \sum_{k=1}^{2} \sum_{j=1}^{3} b_{kj} y_{kj}$$

$\boxed{2 \text{ min}}$ 1

(capacity restr.) $\qquad \sum_{k=1}^{2} x_{ik} \leq s_i \qquad i = 1, \dots, 4$

(demand) $\qquad \sum_{k=1}^{2} y_{kj} \geq d_j \qquad j = 1, 2, 3$

(flow balance) $\qquad \sum_{i=1}^{4} x_{ik} = \sum_{j=1}^{3} y_{kj}$

$x_{ik}, y_{kj} \geq 0 \qquad i = 1, \dots, 4$

$k = 1, 2$

$j = 1, 2, 3$

Ex 5, Consider constraints

$$a^T x \le b \qquad (\mathrm{I})$$
$$c^T x \le d \qquad (\mathrm{II})$$
$$x \le \mathbb{1}^n$$
$$x \ge 0^n$$

$$a, c \in \mathbb{B}^n \qquad b, d \in \mathbb{R}_+$$

assume that only one of (I) and (II) must be fulfilled. Formulate this as an IP.

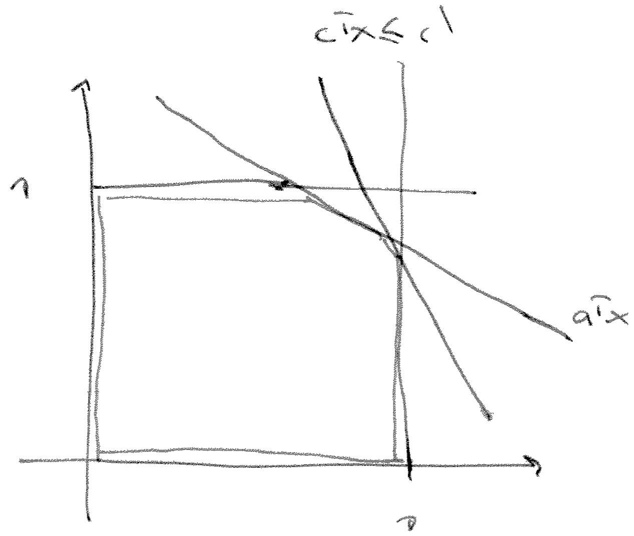
$\boxed{2 \text{ min}}$

$$a^T x \le b + z \qquad (\mathrm{I})$$

$$c^T x \le d + (1-z)\, n \qquad (\mathrm{II})$$

$$0^n \le x \le \mathbb{1}^n$$

$$z \in \{0, 1\}$$



Ex.6 Diet problem

Formulate an LP that minimizes cost of food but satisfies the requirements on ~~[illegible]~~ nutrients.

variables:

x_i – amount of food of type i $i=1,2$

constants:

d_j req. amount of ~~[illegible]~~ nutrient j

c_i cost of food type i /unit

a_{ij} amount of nutrient j in one unit of food type i.

$$\min \sum_{i=1}^{2} c_i x_i$$

$$\sum_{i=1}^{2} a_{ij} x_i \geq d_j \qquad j=1,\dots,3$$

$$x_i \geq 0 \qquad i=1,2$$

$$x^* = \begin{pmatrix} 3.75 \\ 0 \end{pmatrix} \text{kg}$$

$$z^* = \$2.25$$

where $c = [0.6 \quad 0.35]^T$

$d = [8 \quad 15 \quad 3]^T$

$$a = \begin{bmatrix} 5 & 4 & 2 \\ 7 & 2 & 1 \end{bmatrix}$$