Ex 2.1 , convex functions

Assume that $g: \mathbb{R}^n \longrightarrow \mathbb{R}$ is convex

show that:

a) $f(x) = -\ln(-g(x))$ is convex on

$S = \{ x \in \mathbb{R}^n \mid g(x) < 0 \}$

b) $f(x) = \dfrac{1}{\ln(-g(x))}$ is convex on

$S = \{ x \in \mathbb{R}^n \mid g(x) < -1 \}$

Definition f is convex on $S \Leftrightarrow$
For $x_1, x_2 \in S$, $\lambda x_1 + (1-\lambda) x_2 \in S$, $\lambda \in (0,1)$

$\Rightarrow f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$

a) g convex $\Rightarrow$ S convex :

for $x_1, x_2 \in S$ $\quad g(\lambda x_1 + (1-\lambda) x_2) \leq \lambda \underset{0}{\underset{\wedge}{g(x_1)}} + (1-\lambda) \underset{0}{\underset{\wedge}{g(x_2)}} \underset{0}{\wedge}$

hence $\lambda x_1 + (1-\lambda) x_2 \in S \quad \forall \lambda \in (0,1)$

show that f is convex on S:

let $x_1, x_2 \in S$ and $\lambda \in (0,1)$

$$f(\lambda x_1 + (1-\lambda)x_2) = -\ln(-g(\lambda x_1 + (1-\lambda)x_2))$$

$$\underset{\substack{\uparrow \\ h(y) = -\ln(-y)}}{=} h(g(\lambda x_1 + (1-\lambda)x_2)) \underset{\substack{\uparrow \\ h \text{ incr. on } \{y : y<0\} \ ① \\ S \text{ convex}}}{\leq} h(\lambda g(x_1) + (1-\lambda)g(x_2))$$

$$\underset{\substack{\uparrow \\ ② \ h \text{ convex}}}{\leq} \lambda h(g(x_1)) + (1-\lambda) h(g(x_2)) = \lambda f(x_1) + (1-\lambda) f(x_2)$$

① h increasing on $\{y : y < 0\}$

$$\frac{d}{dy} - \ln(-y) = \frac{-1}{-y} \cdot (-1) = -\frac{1}{y} > 0$$

$$\Rightarrow y_1 \leq y_2 \Rightarrow h(y_1) \leq h(y_2)$$

since g convex $g(\lambda x_1 + (1-\lambda)x_2) \leq \lambda g(x_1) + (1-\lambda) g(x_2)$

② h convex on $\{y : y < 0\}$

$$\frac{d^2 h}{dx^2} = \frac{d}{dx} - \frac{1}{y} = \frac{1}{y^2} > 0 \Rightarrow h \text{ convex}$$

b) S is convex (similar proof as in a)

$x_1, x_2 \in S \Rightarrow \lambda x_1 + (1-\lambda) x_2 \in S$

2 min
now proceed?

For $x_1, x_2 \in S$ $\lambda \in (0,1)$

$$f(\lambda x_1 + (1-\lambda) x_2) = h(g(\lambda x_1 + (1-\lambda) x_2))$$

$$h(y) = \frac{1}{\ln(-y)}$$

$$\leq h(\lambda g(x_1) + (1-\lambda) g(x_2)) \leq$$

g convex ①
h increasing

② h convex

$$\lambda h(g(x_1)) + (1-\lambda) h(g(x_2)) = \lambda f(x_1) + (1-\lambda) f(x_2)$$

① $$\frac{d}{dy} h(y) = \frac{d}{dy} (\ln(-y))^{-1} = (-1)(\ln(-y))^{-2} \cdot \frac{1}{-y} - 1$$

$$= \frac{-1}{y(\ln(-y))^2} > 0 \qquad y < -1$$

② $$\frac{d^2}{dy^2} h(y) = \frac{d}{dy} - (y^{-1} \ln(y)^{-2}) =$$

$$= y^{-2}(\ln(-y))^{-2} + -y^{-1}(\ln(-y))^{-3}(-2)\frac{1}{-y}(-1)$$

$$= \frac{1}{y^2 \ln(y)} + \frac{2}{y^2 \ln(-y)^3} = \frac{\ln(-y) + 2}{y^2 \ln(-y)^3} > 0$$

$y < -1$

$\Rightarrow h(y)$ convex.

Ex 2.2

(convex problems)

Def

$$\min f(x)$$
$$\text{s.t. } x \in S$$

is a convex problem $\Leftrightarrow$ f convex S convex

is

$$\max \; -(x_1^2 + \dots + x_n^2)$$

subj. to

$$-\frac{1}{\ln(-g(x))} \geq 0$$

$$d^T x = 2$$

$$g(x) \leq -2$$

$$x \geq 0$$

a convex problem?

1) write the problem in a standard way

the problem is equivalent to

$$\min \quad x_1^2 + \dots + x_n^2 = f(x)$$

$$g_1(x) = 1/\ln(-g(x)) \leq 0$$

$$g_2(x) = d^T x - 2 \leq 0$$

$$g_3(x) = -d^T x + 2 \leq 0$$

$$g_4(x) = g(x) + 2 \leq 0$$

$$g_5(x) = -x \leq 0$$

f is convex since: $\nabla^2 f = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2 \end{bmatrix}$

$\Rightarrow$ all eigenv. of Hessian are ≥ 0

$\Rightarrow$ Hessian pos. def.

$\Rightarrow$ f convex.

$g_1(x)$ is convex (see 1b)

g_2, g_3, g_4, g_5 are obviously convex

$\Rightarrow$ $S = \{x : g_i(x) \leq 0 \quad i=1,\dots,5\}$ is convex

$\therefore$ problem is convex.

Ex 2.3

Is

$$\min \quad x_1 \ln x_1$$

$$\text{subj. to} \quad x_1^2 + x_2^2 \geq 1 \qquad (1)$$

$$2x_1 \geq 1 \qquad (2)$$

$$(x_1-2)^2 + (x_2-2)^2 \leq 1 \qquad (3)$$

$$x_1, x_2 \geq 0 \qquad (4)$$

a convex problem?

$\boxed{2\text{min}}$

How do we proceed?

The problem is equivalent to:

$$\min \quad x_1 \ln x_1$$

$$\text{subj. to} \quad -x_1^2 - x_2^2 + 1 \leq 0 \qquad (1)$$

$$-2x_1 + 1 \leq 0 \qquad (2)$$

$$(x_1-2)^2 + (x_2-2)^2 - 1 \leq 0 \qquad (3)$$

$$-x \leq 0$$

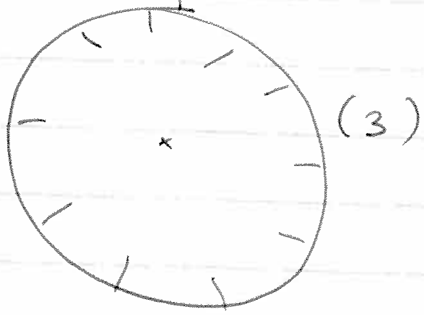
Let $g(x) = -x_1^2 - x_2^2 + 1$

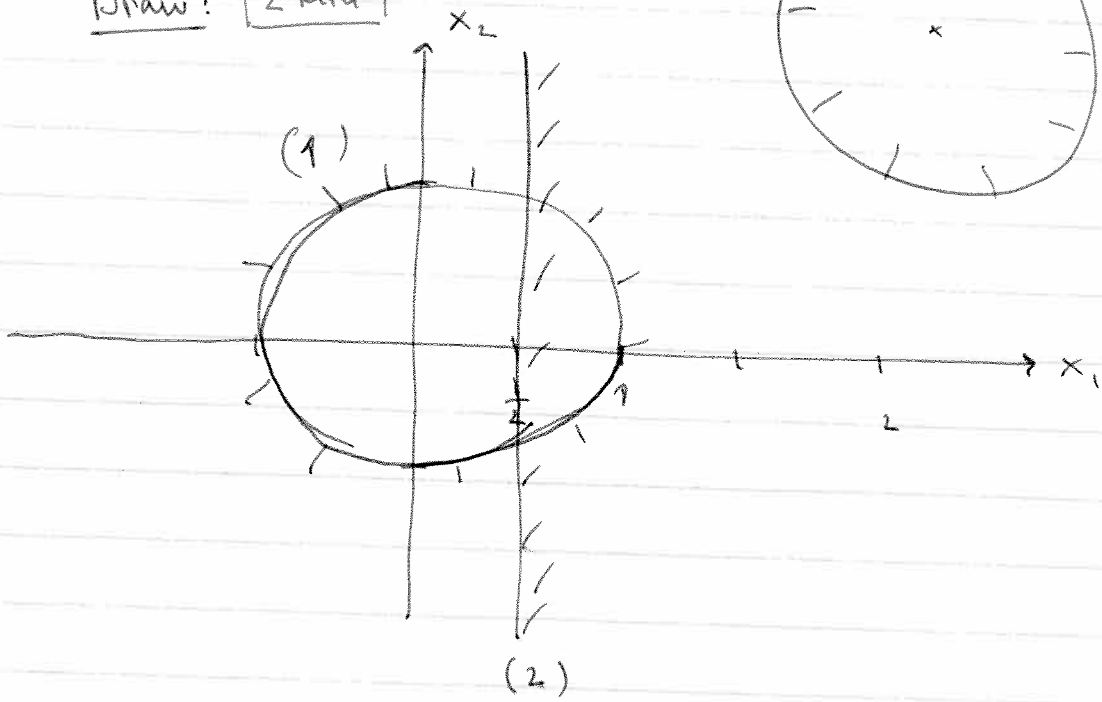
Note that g is non-convex

$$\nabla g = \begin{bmatrix} -2x_1 \\ -2x_2 \end{bmatrix} \qquad \nabla^2 g = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{not pos. def.}$$

Does this imply that the problem is non-convex?

<u>Draw:</u> $\boxed{\text{2 min}}$





we can drop (1) & (2)!

$$\min \quad x_1 \ln x_1 = f(x)$$

$$g(x) = (x_1-2)^2 + (x_1-2)^2 \leq 1$$

$$\nabla^2 g = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$ pos det $\Rightarrow$ g convex

$$\nabla f = \begin{bmatrix} \ln x_1 + x_1 \cdot \frac{1}{x_1} \\ 0 \end{bmatrix}$$

$$\nabla^2 f = \begin{bmatrix} \frac{1}{x_1} & 0 \\ 0 & 0 \end{bmatrix}$$ pos det. since $x_1 \geq \frac{3}{2}$ acc to (2)

$\Rightarrow f$ convex

$\boxed{\therefore \text{ convex problem}}$

$\boxed{\text{theoretical}}$

Ex 2.4 (convexity of polyhedras)

$A \in \mathbb{R}^{m \times n}$ $\qquad b \in \mathbb{R}^m$

show that $P = \{ x \in \mathbb{R}^n \mid Ax \leq b \}$ is convex

Def $S \subseteq \mathbb{R}^n$ convex $\Leftrightarrow$ $x_1, x_2 \in S$

$\Rightarrow \lambda x_1 + (1-\lambda) x_2 \in S \quad \forall \lambda \in (0,1)$

$x_1, x_2 \in P \Rightarrow Ax_1 \leq b$ and $Ax_2 \leq b$

$A(\lambda x_1 + (1-\lambda) x_2) \leq \lambda A x_1 + (1-\lambda) A x_2 \leq \lambda b + (1-\lambda) b = b$

$\Rightarrow \lambda x_1 + (1-\lambda) x_2 \in P \qquad \blacksquare$

$\boxed{\text{scip}}$ $\boxed{\text{theoretical}}$

Ex 2.5 (Farkas lemma)

$$P = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{n+m} \;\middle|\; Ax + By \geq c \,;\, x \geq 0^n \,;\, y \geq 0^m \right\}$$

where $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times m}$ and pos. det.

and $c \in \mathbb{R}^m$.

An Author of a paper assumed that P is compact. Show that the only compact set of type P is the empty set!

Recall Farka's lemma:

$$\begin{array}{ll} Ax = b & \\ x \geq 0 & \text{(I)} \end{array} \qquad \begin{array}{ll} A^T \pi \leq 0 & \\ b^T \pi > 0 & \text{(II)} \end{array}$$

One and only one of the systems (I) and (II) has a solution.

P can be written as

$$Ax + By - Iz = c$$

$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$

by introducing z as slackvariables.

Denote this polyhedron by P'.

if $\exists \begin{pmatrix} x \\ y \end{pmatrix} \in P \Rightarrow \exists \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in P'$ some z.

consider

$$\left.\begin{aligned} &\begin{pmatrix} A & B & -I \\ & e^T & \end{pmatrix} \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &\xi \geq 0 \end{aligned}\right\} (I)$$

when $e^T = (1\ 1\ 1 \dots 1)^T$

If (I) has a solution and $\exists \begin{pmatrix} x \\ y \end{pmatrix} \in P$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \alpha \xi \in P' \quad \forall \alpha \geq 0 \Rightarrow P'$ unbounded since P compact $\Leftrightarrow$ closed + bounded

$\Rightarrow$ if $\exists \begin{pmatrix} x \\ y \end{pmatrix} \in P$ then ~~(II) has a sol.~~ (I) has <u>no</u> solution

Farkas lemma $\Rightarrow$ (II) has a solution

$$\left.\begin{array}{l} \underbrace{\begin{pmatrix} A^T & 1 \\ B^T & e \\ -I & 1 \end{pmatrix} \begin{pmatrix} \pi \\ \mu \end{pmatrix} \leq 0} \qquad \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \pi \\ \mu \end{pmatrix} > 0}_{\Rightarrow \mu > 0} \end{array}\right\} \text{(II)}$$

$\Rightarrow$ $A^T \pi + e\mu \leq 0$ (1)

$B^T \pi + e\mu \leq 0$ (2)

$-\pi + e\mu \leq 0$ (3)

(3) $\Rightarrow$ $e\mu \leq \pi$ $\Rightarrow$ $0 < \mu \leq \pi_i \quad \forall i$

mult. (2) by π^T:

$$\underbrace{\pi^T B \pi}_{\geq 0} + \underbrace{\pi^T e}_{> 0} \underbrace{\mu}_{> 0} \leq 0 \qquad \text{contradiction!}$$

(B pos def.)

$\therefore \nexists \begin{pmatrix} x \\ y \end{pmatrix} \in P$ $\blacksquare$

Ex 2.6

$$x_1 + x_2 \le 2$$

$$x_2 \le 1$$

$$x_3 \le 2$$

$$x_2 + x_3 \le 2$$

a) Is $x^1 = (1 \; 1 \; 0)^T$ an extremepoint?

b) Is $x^2 = (1 \; 1 \; 1)^T$ — " — ?

Thm 3.17

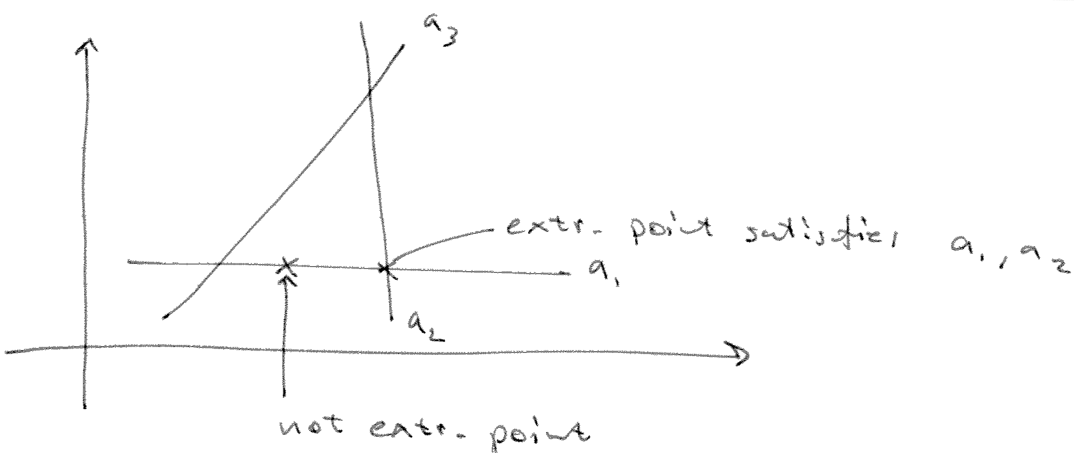
$$\tilde{x} \in P = \{ x \in \mathbb{R}^n \mid Ax \le b \}$$

$A \in \mathbb{R}^{m \times n}$ and $\operatorname{rank} A = n$ and $b \in \mathbb{R}^m$.

Let $\tilde{A}\tilde{x} = \tilde{b}$ be the part of $A\tilde{x} \le b$

which is fulfilled with equality.

Then $\tilde{x}$ is an extremepoint $\iff$ $\operatorname{rank} \tilde{A} = n$



$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{matrix} \quad \sim \quad \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}$$

-1 $\quad -1$

3 lin. indep colons $\Rightarrow$ rank $A = 3$ ok

a) $x^1 = (1 \;\; 1 \;\; 0)^T$ $\qquad \tilde{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \tilde{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

rank $\tilde{A} < 3 \Rightarrow x^1$ not extr. point

$\boxed{\text{2min}}$

b) $x^2 = (1 \;\; 1 \;\; 1)^T$ $\qquad \tilde{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \tilde{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

rank $\tilde{A} = 3 \Rightarrow x^2$ is an extr. point.

theoretical

Ex 2.7 $A \in \mathbb{R}^{m \times n}$ rank $A = m$ $b \in \mathbb{R}^m$

Show that if $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq \mathbb{0}^n\}$

has a feas. solution $\Rightarrow$ P has an extr. point

$x \in P$ ass. that $x = (x_1, \ldots, x_k, 0, \ldots, 0)^T$

where $x_j > 0$ $j = 1, \ldots, k$

if $a_1, \ldots, a_k$ lin. indep $\Rightarrow$ x extr. point

$$x = \begin{pmatrix} B^{-1} b \\ \mathbb{0} \end{pmatrix} \qquad B = \begin{pmatrix} | & & | & | & & | \\ a_1 & \ldots & a_k & a_{k+1} & \ldots & a_m \\ | & & | & | & & | \end{pmatrix}$$

if not $\Rightarrow \exists \lambda_1, \ldots, \lambda_k$ $\sum_{j=1}^{k} \lambda_j a_j = 0$

where some $\lambda_\ell > 0$

Let $\alpha > 0$ s.t. $\alpha = \min\limits_{1 \leq j \leq k} \left\{ \frac{x_j}{\lambda_j} : \lambda_j > 0 \right\} = \frac{x_i}{\lambda_i}$

consider $x' = \begin{cases} x_j - \alpha \lambda_j & j = 1, \ldots, k \\ 0 & j = k+1, \ldots, n \end{cases}$

we have $x'_i = 0$ i.e. x' has at least one additional zero entry.

$$Ax' = Ax - \alpha \sum_{j=1}^{k} \lambda_j a_j = Ax = b$$

$\Rightarrow x' \in P$

- if $a_1, \dots, a_{k-1}$ is linear indep $\Rightarrow x'$ extr. point

- if not iterate. ☒

scip | theoretical

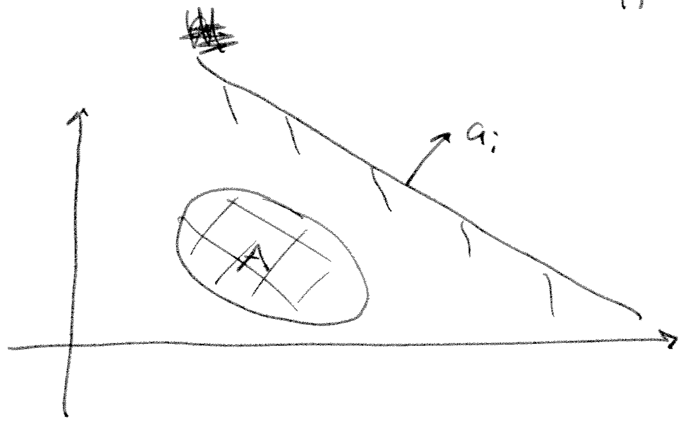
Ex 2.8

A closed + convex

show $A = \bigcap_{A \subseteq B_i} B_i$ där $B_i = \{x \in \mathbb{R}^n \mid a_i^T x \leq b_i\}$

$\uparrow$

i.e. halfplanes



Let $C = \bigcap_{A \subseteq B_i} B_i$

Assume that $x \in A \Rightarrow x \in B_i \quad \forall B_i \supseteq A$

$\Rightarrow x \in C = \bigcap_{A \subseteq B_i} B_i$

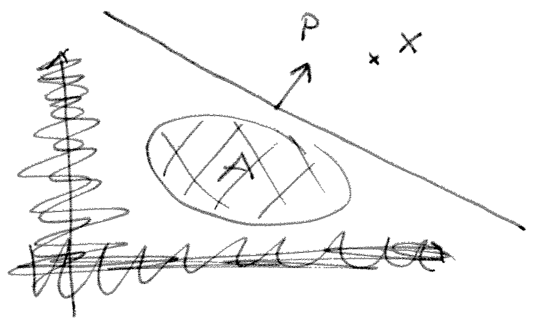
i.e. $A \subseteq C$

Assume $x \in C \quad x \notin A$

Thm 3.24 (sep. thm) $\Rightarrow \exists\, p, \alpha$ s.t.

A convex + closed

$p^T x > \alpha$ and $p^T y \leq \alpha \quad \forall y \in A$



but then $\tilde{B} = \{x \in \mathbb{R}^n \mid p^T x \leq \alpha\}$ is

a halfplane and $A \subseteq \tilde{B}$

since $x \notin \tilde{B}$ we have a contradiction as $x \in C = \bigcap_{A \subseteq B_i} B_i$ by ~~definition~~

$\therefore \; x \in C \Rightarrow x \in A$

i.e. $C \subseteq A$

$\therefore \; A = C \quad \blacksquare$

Does this imply that A is a polyhedron ?
No! The number of halfplanes can be infinite.