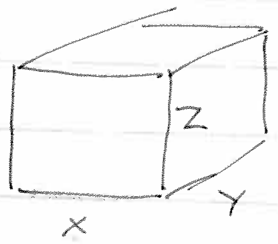
(Unconstrained)

Ex 3.1 Find the rectangular parallelepiped of unit volume and minimal surface area.



$V(x) = xyz$

$A(x) = 2xy + 2xz + 2yz$

(P) $\min A(x)$
$V(x) = 1$
$x \geq 0$

$\underbrace{A \text{ continuous + coersive} \quad \text{closed set}}$

Thm 4.7 (Weirstr. thm)
$\Rightarrow \exists$ opt. solution

$V(x) = xyz = 1 \iff z = \frac{1}{xy}$ if $x, y \neq 0$

$\min \; 2xy + 2\frac{1}{y} + 2\frac{1}{x} \quad f(x,y) \quad (P')$

$x, y > 0$

P' equivalent to P $\Rightarrow$ P' has an opt. solution

If P' has an opt. solution, then it is a local optimum on an open set.
(Or thm 4.23 to open set)

Thm 4.14 (necessary opt. condition) (works on all open sets)

$$\left.\begin{array}{l}\text{local opt}\\ \text{open set}\end{array}\right\} \Rightarrow \nabla f(x) = 0$$

$$\nabla f = \begin{pmatrix} 2y - 2\frac{1}{x^2} \\ \\ 2x - 2\frac{1}{y^2} \end{pmatrix}$$

$$\begin{cases} 2y - 2\frac{1}{x^2} = 0 \\ \\ 2x - 2\frac{1}{y^2} = 0 \end{cases}$$

$x, y \neq 0$

$$\Leftrightarrow \begin{cases} 2yx^2 = 2 \\ \\ 2xy^2 = 2 \end{cases} \Leftrightarrow \begin{array}{l} yx^2 = 1 \\ \\ xy^2 = 1 \end{array}$$

$$\cancel{\Leftrightarrow} \quad \Rightarrow x = \frac{1}{y^2} \Rightarrow \frac{1}{y^3} = 1 \Rightarrow y = 1$$

$$x = 1$$

$(1,1)$ only point that satisfies $\nabla f(x) = 0$

$\Rightarrow$ it is the global opt of P'.

$$\boxed{\begin{array}{l} x = 1 \\ y = 1 \\ z = 1 \end{array}}$$

(Unconstrained)

Ex 3.2 (1) $\min_{x_1, x_2} f(x) = \frac{3}{2}(x_1^2 + x_2^2) + (1+a)x_1x_2 - (x_1+x_2) + b$

Find all the values of a and b s.t. (1) has an unique opt. solution.

Thm 4.17 necessary C^2-case

$$x^* \text{ loc min} \Rightarrow \begin{cases} \nabla f(x^*) = 0 \\ \nabla^2 f(x^*) \text{ pos. semi definite} \end{cases}$$

Thm 4.17 sufficient C^2 case

$$\left.\begin{array}{l} \nabla f(x^*) = 0 \\ \nabla^2 f(x^*) > 0 \end{array}\right\} \Rightarrow x^* \text{ strict local min}$$

$$\nabla f(x) = \begin{pmatrix} 3x_1 + (1+a)x_2 - 1 \\ 3x_2 + (1+a)x_1 - 1 \end{pmatrix}$$

$$\nabla^2 f(x) = \begin{pmatrix} 3 & 1+a \\ 1+a & 3 \end{pmatrix}$$

$\nabla f(x) = 0$

$$\Longleftrightarrow \quad -\frac{(1+a)}{3} \quad \begin{cases} 3x_1 + (1+a)\,x_2 = 1 \\ (1+a)x_1 + 3x_2 = 1 \end{cases}$$

$$\begin{cases} 3x_1 + (1+a)\,x_2 = 1 & (1) \\ \left(3 - \frac{(1+a)^2}{3}\right)x_2 = 1 - \frac{(1+a)}{3} & (2) \end{cases}$$

$$(2) \Rightarrow \quad 9 - (1+2a+a^2)x_2 = 3 - (1+a) = 2-a$$

$$\Longleftrightarrow \quad -(a^2+2a-8)x_2 = -(a-2)$$

$$a^2+2a-8 = (a-2)(a+4)$$

$$\Rightarrow \quad x_2 = \frac{1}{a+4} \qquad a \neq 2 \qquad a \neq -4$$

$$(1) \Rightarrow \quad x_1 = \frac{1 - \frac{(1+a)}{(a+4)}}{3} = \frac{a+4-(1+a)}{3(a+4)} = \frac{1}{a+4}$$

$$a = 2 \qquad (2) \Rightarrow \quad 0 = 1 - \frac{3}{3} = 0$$

$$(1) \Rightarrow \quad x_2 = t \qquad x_1 = \frac{1}{3} - t$$

$$a = -4 \qquad (2) \Rightarrow \quad 0 = 1 - \frac{(-3)}{3} = 2 \qquad \text{no solutions!}$$

Eigenvalues of $\nabla^2 f(x)$ $\boxed{\text{ideas?}}$

$$\begin{vmatrix} 3-\lambda & 1+a \\ 1+a & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - (1+a)^2 = 0$$

$$\iff \lambda^2 - 6\lambda + 9 - (1+a)^2 = 0$$

$$\lambda = 3 \pm \sqrt{9-9+(1+a)^2} = 3 \pm |1+a|$$

$$|1+a| < 3 \Rightarrow \nabla^2 f \text{ pos. def.}$$

$$\iff -3 < 1+a < 3 \iff -4 < a < 2$$

$$x_1 = x_2 = \frac{1}{1+a} \qquad \left.\begin{array}{l} \nabla f(x) = 0 \\ \nabla^2 f(x) \succ 0 \end{array}\right\} \Rightarrow x \text{ strict opt.}$$

$$\left.\begin{array}{l} \text{only one loc} \Rightarrow \text{it is unique local} \\ \text{for } -4 < a \quad f \text{ coercive} \\ \qquad \Rightarrow \exists \text{ global opt. solution} \end{array}\right\} \Rightarrow \exists \text{ unique global}$$

Need this, consider

if $a < -4$ or $a > 2$ $\nabla^2 f(x)$ has neg. eigenvalues

$\Rightarrow$ necessary opt. conditions are not fulfilled

$\Rightarrow$ no opt. solutions.

$a = -4$ $\nabla f(x) = 0$ never fulfilled $\Rightarrow$ no opt. sol.

$a = 2$, $\Rightarrow \nabla f(x^*) = 0$ $\nabla^2 f(x^*)$ pos semi def

$$x^* = \left(1 - \frac{t}{3}, t\right)$$

f quadratic function

$$f(x) = f(x^*) + \underset{\overset{\shortparallel}{0}}{\nabla f(x^*)}(x - x^*) + \underbrace{(x-x^*)^T \nabla^2 f(x^*)(x-x^*)}_{\geq 0}$$

$$\geq f(x^*) \quad \forall x \in \mathbb{R}^n$$

x^* global opt sol and $\exists$ infinite number of opt. sol.

$\therefore$ Only for $-4 < a < 2$ do unique global opt. sol. exist and $x_1^* = x_2^* = \frac{1}{a+4}$.

$\boxed{\text{scip}}$

Ex 3.3

An $n \times n$ symetric matrix

Let $x \in \mathbb{R}^n$ s.t. $x \neq 0^n$

consider $g(x) = \dfrac{x^T A x}{x^T x}$

$$\min_{x \neq 0} g(x)$$

find all stationary points + global min.

stationary point : $\nabla g(x) = 0$

$$\nabla x^T A x = 2Ax$$

$\uparrow$

A symetric

$$\nabla x^T x = 2x$$

$$\frac{d}{dx_k} g(x) = \frac{d}{dx_k}(x^T A x)\frac{1}{x^T x} + x^T A x - \frac{1}{(x^T x)^2}\frac{d}{dx_k} x^T x$$

$$\nabla g(x) = \frac{\nabla(x^T A x)}{x^T x} - \frac{x^T A x}{(x^T x)^2}\nabla(x^T x)$$

$$= \frac{2Ax}{x^T x} - 2x\frac{x^T A x}{(x^T x)^2} = \frac{2}{x^T x}\left(Ax - g(x)x\right) = 0$$

$\because$ $\nabla g(x) = 0 \iff x^T x \neq 0$ and $Ax = g(x)\, x$

satisfied if and only if x eigenv. and $g(x)$ eigenv. corr. to x.

Ass x eigenv. $g(x) = \frac{x^T A x}{x^T x} = \frac{x^T \lambda x}{x^T x} = \lambda$ ok!

stationary points: eigenvectors of A and $g(x_i) = \lambda_i$

$\min\limits_{x \neq 0} \frac{x^T A x}{x^T x}$ (P) is equivalent to

$\min\limits_{\|x\|=1} x^T A x$ (P')

$\left.\begin{array}{ll} x^T A x & \text{continuous on } \|x\| = 1 \\ \|x\| = 1 & \text{compact} \end{array}\right\} \Rightarrow \exists$ global opt x^*

$\Rightarrow x^*$ global opt in P.

x^* global opt $\Rightarrow$ $\left.\begin{array}{l} \text{- local opt} \\ \text{- open set} \end{array}\right\} \Rightarrow$ stationary point least stationary point $\Leftrightarrow$ least eigenvalue

$\because$ $\min_{x \neq 0} \dfrac{x^T A x}{x^T x}$ is solved by x_i

which is eigenvector to

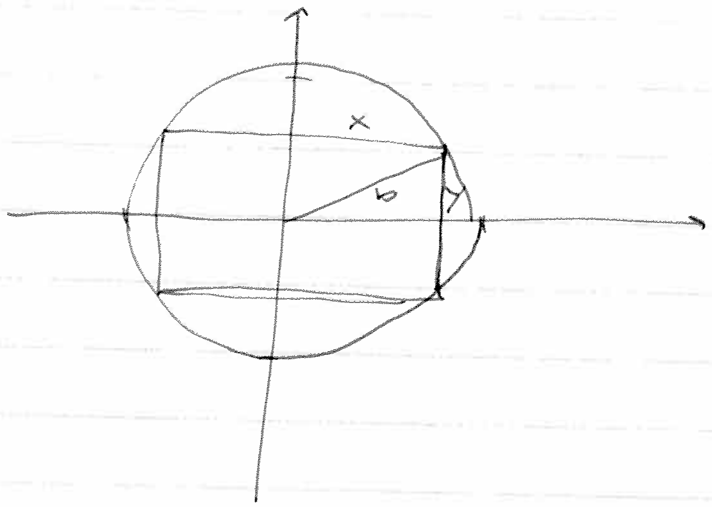
$\min_i \lambda_i$.

(variational ineq.)

Ex 3.4

~~Ex 3.4~~

Show that of all rect. contained in a given circle, the one with maximal area is a square.



max $2x \cdot 2y$
$x^2+y^2 \le b$
$x, y \ge 0$

min $-2x \cdot 2y = f(x)$
$x^2+y^2 \le b$
$x, y \ge 0$

f is cont. on a compact set $\Rightarrow \exists$ global opt.

Thm 4.2.3 (convex feas. set)

$\nabla f(x^*)(x - x^*) \ge 0$ necessary condition
$\forall x \in S$ (variational inequality)

which points satisfy this?

$$\nabla f(x^*) = \begin{pmatrix} -4y^* \\ -4x^* \end{pmatrix}$$

$$\Rightarrow \quad -4y^*(x-x^*) - 4x^*(y-y^*) \geq 0$$

$$\Leftrightarrow \quad y^*(x-x^*) + x^*(y-y^*) \leq 0 \qquad (1)$$

$\forall x, y$ s.t. $x^2+y^2 \leq b \quad x, y \geq 0$

satisfied by $y^* = x^* = 0$

Assume $x^* > y^*$

let $x = y^* \quad y = x^*$

$$\Rightarrow y^*(y^*-x^*) + x^*\underbrace{(x^*-y^*)}_{0} > y^*(y^*-x^*) + y^*(x^*-y^*) = 0$$

i.e. (1) <u>not</u> satisfied similarly $x^* < y^*$ does not satisfy (1)

$$\Rightarrow y^* = x^*$$

Assume $0 \neq y^* = x^* < \frac{b}{\sqrt{2}}$

Let $y = x = \frac{b}{\sqrt{2}}$

$$y^*\left(\frac{b}{\sqrt{2}} - x^*\right) + x^*\left(\frac{b}{\sqrt{2}} - y^*\right) > 0$$

$\Rightarrow$ (1) not satisfied

$\because$ (1) only sat. by

$x^*=0 \quad y^*=0$ and $x^*=y^*=\frac{b}{\sqrt{2}}$

One of these must be global opt.

$f(0)=0 \qquad f\left(\frac{b}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)=\frac{4b^2}{2}=2b^2>0$

maximal area achieved with $x^*=y^*=\frac{b}{\sqrt{2}}$.

Ex 3.5

~~Ex 4~~ (The variational ineq.)

Thm S convex $f \in C^1$

$x^* \in S$ loc opt $\Rightarrow \underbrace{\nabla f(x^*)(x - x^*) \geq 0 \quad \forall x \in S}_{\text{variational ineq.}}$

$S = \{x \in \mathbb{R}^n \mid x \geq 0\}$

Derive the necessary opt. conditions for $\min_{x \in S} f(x)$

Necessary $\nabla f(x^*)(x - x^*) \geq 0 \quad \forall x \in S$

let $x = x_i^* + \varepsilon e_i \quad \varepsilon > 0$

$\Rightarrow \nabla f(x^*)_i \varepsilon \geq 0 \rightarrow \nabla f(x^*) \geq 0$

if $x_i^* > 0 \quad x = x_i^* - \varepsilon e_i$

$-\nabla f(x^*)_i \varepsilon \leq 0 \Rightarrow \nabla f(x_*)_i = 0$

$\therefore$ x^* loc opt $\Rightarrow \begin{cases} x^* \geq 0 \\ \nabla f(x^*)^T x^* = 0 \\ \nabla f(x^*) \geq 0 \end{cases}$