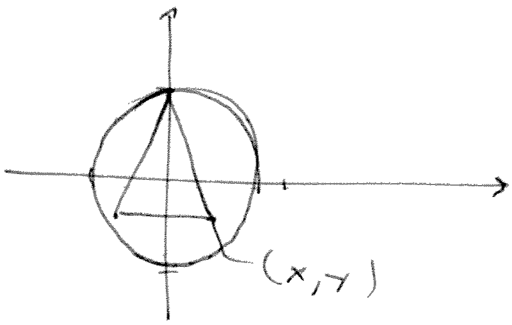
Ex 1 Find largest isosceles, triangle contained in the unit circle



area: $\dfrac{2x(1+y)}{2} = x(1+y)$

$x^2+y^2 \leq 1$

$x, y \geq 0$

$\min f(x) = -x(1+y)$

$g_1(x) = x^2+y^2-1 \leq 0$

$g_2(x) = -x \leq 0$

$g_3(x) = -y \leq 0$

all constr. convex + $\exists$ inner point $\Big\}\Rightarrow$ slater's CQ

so x^* loc opt $\Rightarrow$ KKT (f is nonconvex, so KKT is <u>not</u>

Weierstr $\Rightarrow \exists$ opt sol

~~Ass~~

Ass $x, y > 0$ and $x^2+y^2 < 1$

KKT $\Rightarrow$ $\nabla f(x^*) = 0$

$$\nabla f(x) = \begin{pmatrix} -(1+y) \\ -x \end{pmatrix} = 0 \Rightarrow \begin{matrix} x = 0 \\ y = -1 \end{matrix}$$ infeasible !

no KKT

~~$x \geq 0$ $y = 0$ x^2~~

$x = 0$ $y > 0$ $x^2+y^2 < 1$

KKT $\nabla f(x) + \mu_2 \nabla g_2(x) = 0$

$\mu_2 \geq 0$

$$\nabla g_2(x) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{cases} -(1+y) - \mu_2 = 0 \\ -x = 0 \end{cases} \Rightarrow x = 0$$

$$\mu_2 = -\frac{1}{1+y} < 0$$

no KKT

similarly $x > 0$ $y = 0$ $x^2+y^2 < 1$ has no KKT

$x > 0, \quad y > 0 \qquad x^2 + y^2 = 1$

$\nabla f(x^*) + \mu_1 \nabla g_1(x) = 0$

$\mu_1 \geq 0$

$\nabla g_1(x) = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$

$$\begin{cases} -(1+y) + 2x\mu_1 = 0 & (1) \\ -x + 2y\mu_1 = 0 & (2) \end{cases}$$

$(2) \Rightarrow \mu_1 = \dfrac{x}{2y}$

$(1) \Rightarrow -(1+y) + \dfrac{2x^2}{2y} = 0$

$\Leftrightarrow -(1+y)y + x^2 = 0$

$x^2+y^2=1$

$\Leftrightarrow -y^2 - y - y^2 + 1 = 0$

$\Leftrightarrow y^2 + \frac{1}{2}y - \frac{1}{2} = 0 \qquad y = \begin{cases} -1 \leftarrow \text{infeas.} \\ \frac{1}{2} \end{cases}$

$x = \sqrt{1 - \frac{1}{4}} = \dfrac{\sqrt{3}}{2}$

$f\left(\frac{\sqrt{3}}{2}, 1\right) = -\frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right) = -\dfrac{\sqrt{3} \cdot 3}{4} \simeq -1.299 \cancel{<} 1$

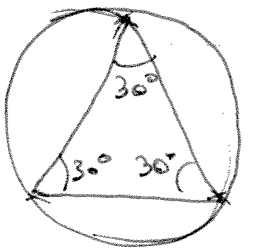
Only points lett:

$x=0 \quad y=0 \qquad f(0,0)=0$

$x=0 \quad y=1 \qquad f(0,1)=0$

$x=1 \quad y=0 \qquad f(1,0)=-1$

$\because$ opt. sol. $x=\frac{\sqrt{3}}{2} \quad y=1$



Ex 2 (KKT - finding opt. sol.)

$$\min \ f(x) = \sum_{j=1}^{n} c_j x_j$$

subj. to $\sum_{j=1}^{n} x_j^2 \leq 1$

$x_j \geq 0 \qquad j = 1, \dots, n$

where $\min_{j=1,\dots,n} c_j < 0$

Find opt. sol.

$\longrightarrow$ $\boxed{2\text{min}}$

convex con. + inner point $\Big\} \Rightarrow$ Slater's CQ

convex problem $\Rightarrow$ suff. KKT

x^* opt $\Leftrightarrow$ x^* KKT

let $g_i(x) = -x_i \qquad i = 1, \dots, n \qquad \nabla g_i(x) = \begin{pmatrix} 0 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{pmatrix}$

$$g_{n+1} = \sum_{j=1}^{n} x_j^2 - 1 \qquad \nabla g_{n+1} = 2 \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\nabla f = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

~~KKT:~~ ~~$\nabla f(x) +$~~

if $g_{n+1}(x) < 0 \Rightarrow \mu_{n+1} = 0$

$$\Rightarrow \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + \sum_{i=1}^{n} \mu_i \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$$

$\mu_i \geq 0 \quad i = 1, \dots, n$

no sol since $\min\{c_i\} < 0$

if $g_{n+1}(x) = 0$

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + 2\mu_{n+1} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} - \sum \mu_i \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0$$

i-th pos

$x_i > 0 \rightarrow \mu_i = 0 \qquad i = 1, \dots, n$

$\mu_i \geq 0 \quad \forall i$

~~conclude that $x_i = 0$ if $c_i > 0$~~

$c_i + 2\mu_{n+1} x_i - \mu_i = 0$

if $x_i > 0 \quad c_i > 0$ no sol. exists

$c_i > 0 \Rightarrow x_i = 0 \Rightarrow \mu_i = c_i$

$c_i < 0 \quad x_i \geq 0$ no sol.

$c_i < 0 \quad x_i > 0 \qquad c_i + 2\mu_{n+1} x_i = 0 \iff x_i = \frac{-c_i}{2\mu_{n+1}} > 0$

$c_i = 0 \quad x_i = 0 \quad \Rightarrow \quad c_i + \mu_i = 0 \quad \mu_i = 0$

$c_i = 0 \quad x_i > 0 \quad \Rightarrow \quad 2\mu_{n+1} x_i = 0 \Rightarrow \mu_{n+1} = 0$

not possible

$\therefore \quad c_i \geq 0 \Rightarrow x_i = 0 \quad \mu_i = c_i$

$\qquad c_i < 0 \Rightarrow x_i = \frac{-c_i}{2\mu_{n+1}} > 0 \quad \mu_i = 0$

$$\sum_{i=1}^{n} x_i^2 = 1 \iff \sum \frac{c_i^2}{4\mu_{n+1}^2} = 1$$

$$\iff \mu_{n+1} = \frac{1}{2}\sqrt{\sum_{i=1}^{n} c_i^2}$$

only KKT point is $x_i = 0$ if $c_i \geq 0$

$$x_i = \frac{-c_i}{\sqrt{\sum c_i^2}} \quad c_i < 0$$

and is optimal.

Ex 3

$$\min \quad f(x) = \sum_{j=1}^{n} c_j x_j^2$$

$$\text{subj. to} \quad \sum_{j=1}^{n} x_j = b$$

$$b, c_j > 0$$

Find opt sol + show unique

$$h(x) = \sum_{j=1}^{n} x_j - b \qquad \text{linear} \Rightarrow \text{Slater CQ}$$

$$\nabla h = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

x loc opt $\Rightarrow$ KKT

$$\nabla f(x) = 2 \begin{pmatrix} c_1 x_1 \\ \vdots \\ c_n x_n \end{pmatrix}$$

$$\nabla^2 f(x) = 2 \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{pmatrix} \qquad c_i > 0 \ \forall i \quad \text{pos. def.}$$

$\therefore$ f str. convex

x^* glob opt $\Leftrightarrow$ x^* KKT

KKT:

$$2\begin{pmatrix} c_1 x_1 \\ \vdots \\ c_n x_n \end{pmatrix} + v\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$2c_i x_i + v = 0$$

$$\Rightarrow x_i = -\frac{v}{2c_i}$$

$$\sum x_i = v \sum -\frac{1}{2c_i} = b$$

$$\Leftrightarrow \quad v = \frac{b}{\sum -\frac{1}{2c_i}}$$

only one KKT $\quad x_i^* = \dfrac{b}{\left(\sum_{j=1}^{n} \frac{1}{2c_j}\right) c_i}$

$\Rightarrow$ x^* only opt sol.

Ex 4

$$\min \ f(x) = 4x_1^2 + 2x_2^2 - 6x_1x_2 + x_1$$

subj. to
$$-2x_1 + 2x_2 \geq 1$$
$$2x_1 - x_2 \leq 0$$
$$x_1 \leq 0$$
$$x_2 \geq 0$$

Is $x = (0 \quad 1/2)^T$ a KKT point?

Conclusion for opt. of x?

$$\min \ f(x)$$
$$g_1(x) = 2x_1 - 2x_2 + 1 \leq 0$$
$$g_2(x) = 2x_1 - x_2 \leq 0$$
$$g_3(x) = x_1 \leq 0$$
$$g_4(x) = -x_2 \leq 0$$

$\boxed{2 \text{ min}}$

active constr. at $\bar{x}$:

$$g_1(\bar{x}) = 0$$
$$g_2(\bar{x}) = -\frac{1}{2} < 0$$
$$g_3(\bar{x}) = 0$$
$$g_4(\bar{x}) = -\frac{1}{2} < 0$$

g_2, g_4 active $\quad \nabla f(x) = \begin{pmatrix} 8x_1 - 6x_2 + 1 \\ 4x_2 - 6x_1 \end{pmatrix} \quad \nabla f(\bar{x}) = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

$$\nabla g_1(x) = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \qquad \nabla g_2(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\bar{x}$ KKT $\Leftrightarrow$ $\exists \mu_1, \mu_2 \geq 0$ s.t.

$$\begin{pmatrix} -2 \\ 2 \end{pmatrix} + \mu_1 \begin{pmatrix} 2 \\ -2 \end{pmatrix} + \mu_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

for instance $\mu_1 = 1 \quad \mu_2 = 0$

$\Rightarrow \bar{x}$ is a KKT point.

$$\nabla^2 f(x) = \begin{pmatrix} 8 & -6 \\ -6 & 4 \end{pmatrix}$$

eigenval. $\begin{vmatrix} 8-\lambda & -6 \\ -6 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 6 \pm \sqrt{40}$

$\Rightarrow \exists$ neg. eigenval.

f not convex $\Rightarrow$ KKT <u>not</u> suff. to conclude something about opt.

Ex 5 $\quad \min f(x) = x_1^2 + 3x_2^2 - x_1$

subj. to $\quad x_1^2 - x_2 \leq 1$

$x_1 + x_2 \geq 1$

is $\bar{x} = (1 \;\; 0)$ a global min?

$\min f(x)$

subj. to $\quad g_1(x) = x_1^2 - x_2 - 1 \leq 0$

$g_2(x) = -x_1 - x_2 + 1 \leq 0$

Is $\bar{x}$ KKT?

$$\nabla f(x) = \begin{pmatrix} 2x_1 - 1 \\ 6x_2 \end{pmatrix} \qquad \nabla f(\bar{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

active constr:

$g_1(\bar{x}) = 1 - 1 = 0$

$g_2(\bar{x}) = -1 + 1 = 0$ $\qquad$ both!

$$\nabla g_1(x) = \begin{pmatrix} 2x_1 \\ -1 \end{pmatrix} \qquad \nabla g_2(x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\nabla g_1(\bar{x}) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} =$$

$$\begin{pmatrix}1\\0\end{pmatrix} + \mu_1\begin{pmatrix}2\\-1\end{pmatrix} + \mu_2\begin{pmatrix}-1\\-1\end{pmatrix} = 0 \qquad \text{no sol. } \mu_1, \mu_2 \geq 0$$

$\Rightarrow \bar{x}$ no KKT

$$\nabla^2 g_1(\bar{x}) = \begin{pmatrix}2 & 0\\0 & 0\end{pmatrix} \quad \text{pos def.}$$

g_2 affine

$\Rightarrow$ conv. constr.

$\tilde{x} = (0 \ 2)^T$ is an inner point $\quad g_1(\tilde{x}) = -3 < 0$

$g_2(\tilde{x}) = -1 < 0$

$\Rightarrow$ Slater $\Rightarrow$ KKT necessary

$\therefore \bar{x}$ is <u>not</u> optimal