Lecture 9: The Simplex method

An algebraic derivation of the pricing step

$$
\begin{aligned}
z^* = \quad & \text{infimum } \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} & = \quad & \text{infimum } \boldsymbol{c}_{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{x}_{\boldsymbol{B}} + \boldsymbol{c}_{\boldsymbol{N}}^{\mathrm{T}}\boldsymbol{x}_{\boldsymbol{N}} \\
& \text{subject to } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, & & \text{subject to } \boldsymbol{B}\boldsymbol{x}_{\boldsymbol{B}} + \boldsymbol{N}\boldsymbol{x}_{\boldsymbol{N}} = \boldsymbol{b}, \\
& \qquad \boldsymbol{x} \geq \boldsymbol{0}^n & & \qquad \boldsymbol{x}_{\boldsymbol{B}} \geq \boldsymbol{0}^m;\ \boldsymbol{x}_{\boldsymbol{N}} \geq \boldsymbol{0}^{n-m}
\end{aligned}
$$

$$
\begin{aligned}
= \boldsymbol{c}_{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{b} + \quad & \text{infimum } [\boldsymbol{c}_{\boldsymbol{N}}^{\mathrm{T}} - \boldsymbol{c}_{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{N}]\boldsymbol{x}_{\boldsymbol{N}} \\
& \text{subject to } \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_{\boldsymbol{N}} \geq \boldsymbol{0}^m, \\
& \qquad \boldsymbol{x}_{\boldsymbol{N}} \geq \boldsymbol{0}^{n-m}
\end{aligned}
$$

- $\boldsymbol{x}_{\boldsymbol{N}} = \boldsymbol{0}^{n-m}$ feasible. Let $\tilde{\boldsymbol{c}}_{\boldsymbol{N}}^{\mathrm{T}} := \boldsymbol{c}_{\boldsymbol{N}}^{\mathrm{T}} - \boldsymbol{c}_{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{N}$
- If *reduced cost* $\tilde{\boldsymbol{c}}_{\boldsymbol{N}} \geq \boldsymbol{0}^{n-m}$ then $\boldsymbol{x}_{\boldsymbol{N}} = \boldsymbol{0}^{n-m}$ is optimal

- If $\tilde{\boldsymbol{c}}_{\boldsymbol{N}} \not\geq \mathbf{0}^{n-m}$ then $\exists j \in N$ with $\tilde{c}_j < 0$. Then the current point $\boldsymbol{x}_{\boldsymbol{N}} = \mathbf{0}^{n-m}$ may be non-optimal
- Generate a feasible descent direction
- Choose one that leads to a neighboring extreme point
- Swap one variable in B for one in N
- Increase one variable in N from zero
- Choose $\jmath^*$ to be among $\arg\operatorname{minimum}_{j \in N} \tilde{c}_j$ (the "incoming" variable)
- We have then decided on the search direction

The basis change

- What is this direction?
- In $\boldsymbol{x_N}$-space: $\boldsymbol{p_N} = \boldsymbol{e}_{\mathrm{J}^*}$ (unit vector)
- In $\boldsymbol{x_B}$-space: $\boldsymbol{x_B} = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x_N} \Longrightarrow$
  $\boldsymbol{p_B} = -\boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{p_N} = -\boldsymbol{B}^{-1}\boldsymbol{N}_{\mathrm{J}^*}$
- So, search direction in $\mathbb{R}^n$:

$$\boldsymbol{p} = \begin{pmatrix} \boldsymbol{p_B} \\ \boldsymbol{p_N} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{B}^{-1}\boldsymbol{N}_{\mathrm{J}^*} \\ \boldsymbol{e}_{\mathrm{J}^*} \end{pmatrix}$$

- Descent? Yes, because $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{p} = \tilde{c}_{\mathrm{J}^*} < 0$!

- Feasible? Must check that $\boldsymbol{Ap} = \mathbf{0}^m$ and that $p_i \geq 0$ if $x_i = 0$, $i \in B$. The first true by construction:
- (a) $\boldsymbol{Ap} = \boldsymbol{B}\boldsymbol{p}_{\boldsymbol{B}} + \boldsymbol{N}\boldsymbol{p}_{\boldsymbol{N}} = -\boldsymbol{B}\boldsymbol{B}^{-1}\boldsymbol{N}_{\jmath^*} + \boldsymbol{N}\boldsymbol{e}_{\jmath^*} = \mathbf{0}^m$
- (b) Suppose that $\boldsymbol{x}_{\boldsymbol{B}} > \mathbf{0}^m$. Then, at least a small step in $\boldsymbol{p}$ keeps $\boldsymbol{x}_{\boldsymbol{B}} \geq \mathbf{0}^m$.

  But if there is an $\imath^*$ with $(\boldsymbol{x}_{\boldsymbol{B}})_{\imath^*} = 0$ *and* $(\boldsymbol{p}_{\boldsymbol{B}})_{\imath^*} < 0$ then it is not a feasible direction
- Must then perform a basis change without moving! A *degenerate basis change*: swap $x_{\jmath^*}$ for $x_{\imath^*}$ in the basis
- Otherwise (and normally), we utilize the unit direction

- Line search? Linear objective; move the maximum step!
- Maximum step: If $\boldsymbol{p_B} \geq \mathbf{0}^m$ there is no finite maximum step! We have found an extreme direction $\boldsymbol{p}$ along which $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ tends to $-\infty$! *Unbounded solution*
- Otherwise: choose a basic variable that first reaches zero (the "outgoing" variable): choose a variable $i \in B$ with minimum in

$$x_{\jmath^*} := \underset{i \in B}{\text{minimum}} \left\{ \frac{(\boldsymbol{B}^{-1}\boldsymbol{b})_i}{(\boldsymbol{B}^{-1}\boldsymbol{N}_{\jmath^*})_i} \,\middle|\, (\boldsymbol{B}^{-1}\boldsymbol{N}_{\jmath^*})_i > 0 \right\}$$

- Done. In the basis, replace $\imath^*$ by $\jmath^*$; goto pricing step
- If $x_{\jmath^*} = 0$ then the above corresponds to a "degenerate basis change"

Computational notes—how do we do all of this?

- Given basis matrix $\boldsymbol{B}$, solve

$$\boldsymbol{B}\boldsymbol{x}_{\boldsymbol{B}} = \boldsymbol{b}$$

- Gives us BFS: $\boldsymbol{x}_{\boldsymbol{B}} = \boldsymbol{B}^{-1}\boldsymbol{b}$
- Pricing step: (a) Solve

$$\boldsymbol{B}^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}_{\boldsymbol{B}} \quad \Longrightarrow \quad \boldsymbol{y}^{\mathrm{T}} = \boldsymbol{c}_{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{B}^{-1}$$

- (b) Calculate $\tilde{\boldsymbol{c}}_{\boldsymbol{N}}^{\mathrm{T}} = \boldsymbol{c}_{\boldsymbol{N}}^{\mathrm{T}} - \boldsymbol{y}^{\mathrm{T}}\boldsymbol{N}$, the reduced cost vector
- Choose the incoming variable, x_{j^*}

- Outgoing variable: Solve

$$\boldsymbol{B}\boldsymbol{p}_{\boldsymbol{B}} = -\boldsymbol{N}_{\jmath^*}$$

- Quotient rule for $(\boldsymbol{B}^{-1}\boldsymbol{b})_i/(-\boldsymbol{p}_{\boldsymbol{B}})_i$ gives outgoing variable, $x_{\imath^*}$, *and* value of the new basic variable, $x_{\jmath^*}$
- Note: Three similar linear systems in $\boldsymbol{B}$! LU factorization of $\boldsymbol{B}$ + three triangular substitutions
- Factorizations can be updated after basis change rather than done from scratch
- LP solvers like Cplex and XPRESS-MP have excellent numerical solvers for linear systems
- Linear systems the bulk of the work in solving an LP

Convergence

- *If all of the basic feasible solutions are non-degenerate, then the Simplex algorithm terminates after a finite number of iterations*
- *Proof*: (Rough argument) Non-degeneracy implies that the step length is > 0; hence, we cannot return to an old BFS once we have left it. There are finitely many BFSs

- Degeneracy: Can actually lead to cycling—the same sequence of BFSs is returned to indefinitely!
- Remedy: Change the incoming/outgoing criteria! Bland's rule: Sort variables according to some index ordering. Take the first possible index in the list. Incoming variable first in the list with the right sign of the reduced cost; outgoing variable the first in the list among the minima in the quotient rule

Initial BFS: Phase I of the Simplex method

- If a starting BFS cannot be found, do the following:
- Suppose $\boldsymbol{b} \geq \mathbf{0}^m$. Introduce *artificial variables* a_i in every row (or rows without a unit column)
- Solve the following Phase-I problem:

$$\begin{aligned} \text{minimize} \quad w = & \quad (\mathbf{1}^m)^{\mathrm{T}}\boldsymbol{a} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{I}^m\boldsymbol{a} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \mathbf{0}^n, \\ & \boldsymbol{a} \geq \mathbf{0}^m \end{aligned}$$

- Possible cases: (a) $w^* = 0$, meaning that $\boldsymbol{a}^* = \mathbf{0}^m$ must hold. There is then a BFS in the *original* problem

- Start Phase-II, to solve the original problem, starting from this BFS
- (b) $w^* > 0$. The optimal basis then has some $a_i^* > 0$; due to the objective function construction, there exists no BFS in the original problem. The problem is then infeasible!
- What to do then? Modelling errors? Can be detected from the optimal solution. In fact, some LP problems are pure feasibility problems