Additional exercises to Exercise 5 - KKT
TMA947 and MMG620 Optimization, first course

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Exercise 1 (easy) Consider the following problem:

$$\begin{aligned} \text{minimize} \quad & -(x_1-1)^2-(x_2-1)^2, \\ \text{subject to} \quad & x_1+x_2 \leq 4, \\ & x_1, x_2 \geq 0. \end{aligned}$$

(a) Draw the feasible set defined by the constraints.
(b) Sketch the level set of the objective function.
(c) Draw the negative gradient of the objective function and the gradients of the active constraints at $\mathbf{x}^0 = (0,4)^T$ and $\mathbf{x}^1 = (0,1)^T$. Are $\mathbf{x}^0$ and $\mathbf{x}^1$ KKT-points?
(d) Show analytically that $\mathbf{x}^0$ and $\mathbf{x}^1$ are KKT-points.
(e) Find all KKT-points by visually analyzing the figure? (*Hint: There are seven of them*)
(f) Which points are global optima?

Exercise 2 (easy) In the figure below, four different functions (a)-(d) are plotted with the constraints $0 \leq x \leq 2$.
(i) Which points in each graph are KKT-points with respect to minimization? Which points are local/global optima?
(ii) The function in graph (a) is $f(x) = 2 - 0.5x$. Find all KKT-points analytically.

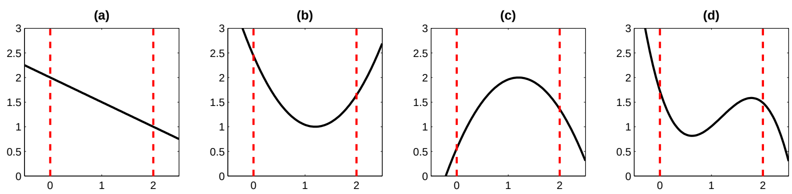

Figure 1: Four different functions (a)-(d). All with constraint $0 \leq x \leq 2$.

Exercise 3 (easy) Consider the problem to

$$\begin{aligned} \text{minimize} \quad & e^{x_1} + x_1^2 x_2, \\ \text{subject to} \quad & x_1 + x_2^2 \geq 4, \\ & x_1, x_2 \geq 0. \end{aligned}$$

a) State the KKT conditions for the problem.
b) Are the KKT conditions satisfied at $(0,2)^T$ and $(1,1)^T$?

Exercise 4 (easy) In which of the following problems are the KKT conditions necessary/sufficient for optimality?

(a)

$$\begin{aligned} \text{minimize} \quad & -2x_1 - 3x_2 + x_3, \\ \text{subject to} \quad & x_1 + 2x_2 + 2x_3 \leq 6, \\ & -6x_1 + 2x_2 - 2x_3 \geq 9, \\ & 2x_1 + 3x_2 + 5x_3 \leq 8, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

(b)

$$\begin{aligned} \text{minimize} \quad & 2x_1^2 + 2x_2^2 - 4x_1 + x_1 x_2, \\ \text{subject to} \quad & x_1 + 2x_2 \leq 6, \\ & 2x_1 - 2x_2 \geq 2, \\ & x_1, x_2 \geq 0. \end{aligned}$$

(c)

$$\begin{aligned} \text{minimize} \quad & 2x_1^2 - (x_2 - 1)^2 + 5, \\ \text{subject to} \quad & x_1^2 + 2x_2^2 \leq 4, \\ & 3x_1 - x_2 \geq 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

(d)

$$\begin{aligned} \text{minimize} \quad & -x_1, \\ \text{subject to} \quad & x_2 - (x_1 - 1)^3 \leq 0, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Exercise 5 (medium) Consider the problem to

$$\begin{aligned} \text{minimize} \quad & -x_1^3 + x_2^2 - 2x_1 x_3^2, \\ \text{subject to} \quad & 2x_1 + x_2^2 + x_3 = 5, \\ & 5x_1^2 - x_2^2 - x_3 \geq 2, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

a) State the KKT conditions for the problem.
b) Verify that the KKT conditions are satisfied at $(1,0,3)^T$

Exercise 6 (medium) Consider the problem:

$$\begin{aligned} &\text{minimize} && f(\mathbf{x}), && \text{(1a)}\\ &\text{subject to} && g_i(\mathbf{x}) \le 0, i = 1,\dots,m, && \text{(1b)} \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1,\dots,m$ are differentiable function. A point $\mathbf{x}^*$ is called a KKT-point if it is feasible in (1), Abadie's constraint qualification holds at $\mathbf{x}^*$ and there exists a vector $\boldsymbol{\mu} \in \mathbb{R}^m$ such that

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} \mu_i \nabla g_i(\mathbf{x}^*) = \mathbf{0}^n, \qquad \text{(2a)}$$

$$\mu_i g_i(\mathbf{x}^*) = 0,\ i = 1,\dots,m, \qquad \text{(2b)}$$

$$\boldsymbol{\mu} \ge \mathbf{0}^m \qquad \text{(2c)}$$

Which of the following statements are true?

(a)

$$\mathbf{x}^* \text{ global minima of (1)} \Rightarrow \mathbf{x}^* \text{ is a KKT point.}$$

(b)

$$\left.\begin{array}{r} \mathbf{x}^* \text{ local minima of (1)} \\ \text{CQ holds at } \mathbf{x}^* \end{array}\right\} \Rightarrow \mathbf{x}^* \text{ is a KKT point}$$

(c)

$$\left.\begin{array}{r} \mathbf{x}^* \text{ is a KKT point} \\ \text{CQ holds at } \mathbf{x}^* \end{array}\right\} \Rightarrow \mathbf{x}^* \text{ local minima of (1)}$$

(d)

$$\left.\begin{array}{r} \mathbf{x}^* \text{ global minimum of (1)} \\ \text{problem (1) is convex} \end{array}\right\} \Rightarrow \mathbf{x}^* \text{ is a KKT point}$$

(e)

$$\left.\begin{array}{r} \mathbf{x}^* \text{ is a KKT point} \\ \text{problem (1) is convex} \end{array}\right\} \Rightarrow \mathbf{x}^* \text{ is global minimun of (1)}$$

Exercise 7 (medium) Consider the problem:

$$\begin{aligned} &\text{minimize} && -2(x_1-2)^2 - x_2^2,\\ &\text{subject to} && x_1^2 + x_2^2 \le 25\\ &&& x_1 \ge 0 \end{aligned}$$

(a) Does any constraint qualification hold?
(b) Find all KKT-points.
(c) Find the global minima.

Exercise 8 (medium) Consider the problem:

$$\begin{aligned} &\text{minimize} && 4x_1^2 + 2x_2^2 - 6x_1x_2 + x_1,\\ &\text{subject to} && -2x_1 + 2x_2 \ge 1,\\ &&& 2x_1 - x_2 \le 0,\\ &&& x_1, x_2 \ge 0. \end{aligned}$$

Is $\mathbf{x} = (0, 1/2)^T$ a KKT point? Can you draw any conclusions from this regarding the optimality of $\mathbf{x}$?