Solutions to exam May 28, 2003

TMA 946 / MAN 280 Applied optimization

1. a) By introducing the slack variables x_5, x_6 and x_7 we get

$$
\begin{array}{llrrrrrrrll}
\text{minimize} & z = & x_1 & -2x_2 & -4x_3 & +4x_4 & & & & & \\
\text{subject to} & & & -x_2 & +2x_3 & +x_4 & +x_5 & & & = & 4, \\
 & & -2x_1 & +x_2 & +x_3 & -4x_4 & & +x_6 & & = & 5, \\
 & & x_1 & -x_2 & & +2x_4 & & & +x_7 & = & 3, \\
 & & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 & \geq & 0.
\end{array}
$$

Let $x_B = [x_5, x_6, x_7]$ and $x_N = [x_1, x_2, x_3, x_4]$ be the initial basic and nonbasic vector respectively. The reduced costs of the nonbasic variables then become

$$c_N^T - c_B^T B^{-1} N = [1, -2, -4, 4],$$

and thus x_3 is the entering variable. Further, we have

$$B^{-1}b = [4, 5, 3]^T,$$
$$B^{-1}N_3 = [2, 1, 0]^T,$$

which gives

$$\underset{j,\,(B^{-1}N_3)_j > 0}{\arg\min} \frac{(B^{-1}b)_j}{(B^{-1}N_3)_j} = 1,$$

so x_5 is the leaving variable. The new basic and nonbasic vectors are $x_B = [x_3, x_6, x_7]$ and $x_N = [x_1, x_2, x_5, x_4]$, and the reduced costs of the nonbasic variables become

$$c_N^T - c_B^T B^{-1} N = [1, -4, 2, 6],$$

so x_2 is the entering variable, and

$$B^{-1}b = [2, 3, 3]^T,$$
$$B^{-1}N_2 = [-1/2, 3/2, -1]^T,$$

which gives

$$\underset{j,\,(B^{-1}N_2)_j > 0}{\arg\min} \frac{(B^{-1}b)_j}{(B^{-1}N_2)_j} = 2,$$

and thus x_6 is the leaving variable. The new basic and nonbasic vectors become $x_B = [x_3, x_2, x_7]$ and $x_N = [x_1, x_6, x_5, x_4]$, and the reduced costs of the nonbasic variables are

$$c_N^T - c_B^T B^{-1} N = [-13/3, 8/3, 2/3, -6],$$

so x_4 is the entering variable and

$$B^{-1}b = [3, 2, 5]^T,$$
$$B^{-1}N_4 = [-1, -3, -1]^T,$$

which means that the problem is unbounded and the objective function diverges to $-\infty$ along the extreme half-line given by

$$x(t) = [x_1, x_2, x_3, x_4] = [0, 2, 3, 0] + t[0, 3, 1, 1], \quad t \geq 0. \tag{1}$$

b) The desired feasible solution can be found along the half-line given by (1). More precisely,

$$c^T x(t) = 0 - 2 \cdot (2 + 3t) - 4 \cdot (3 + t) + 4t = -6t - 16,$$

so by choosing $t = 67$ we get the feasible solution

$$x(67) = [x_1, x_2, x_3, x_4] = [0, 203, 70, 67],$$

and since $c^T x(67) = -418$ we are done.

2a) Take $X_1 = [1,2]$ and $X_2 = [3,4]$. The union of the two intervals clearly is not convex.

b) For each X_i, let $x^i \in X_i$ be one type of constraint. In order to have x belonging to the union of the sets X_i we define an indicator $\delta_i \in \{0,1\}$, for each set. The resulting constraints are as follows:

$$x = \sum_{i=1}^{m} \delta_i x^i$$

$$x^i \in X_i \;, \quad i = 1, \dots, m$$

$$\sum_{i=1}^{m} \delta_i = 1 \,,$$

$$\delta_i \in \{0,1\}, \quad i = 1, \dots, m$$

When δ_i, $i = 1, \dots, m$ is fixed to a feasible value, we have selected one of the sets X_i for which $x = x^i$ holds. Clearly, then, the problem to minimize $f(x)$ subject to the above constraints is a mixed-integer linear program according to the definition.

3.

Variables:

x_1: Bought unlaminated particle-board
x_2: Bought laminated particle-board
Y_1: No of Kalle made and sold
Y_2: No of Billy made and sold

maximize: $x_1 P_1 + x_2 P_2 - b x_1 - c x_2$ (income - expenses)

subject to: $Y_1 \leq k$ (limit on demand of Kalle)

$x_1 + x_2 \geq a_1 Y_1 + a_2 Y_2$: (We must have enough particle-board)

$e_1 Y_1 + e_2 Y_2 \leq 280 \cdot 60$: (We have 280·60 minutes of cutting time)

$f Y_2 + g_1 Y_1 + g_2 Y_2 \leq 120 \cdot 16 \cdot 60$: (We have 120·16·60 minutes of packing and assembly time available)

$x_1 \leq L$ (limit on how much we may laminate ourself)

$0 \leq x_1, x_2, Y_1, Y_2$ No negative production

Question 4.

a) We start from $(x_0, y_0) = (2, 1)$

$$\nabla f(x_0, y_0) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

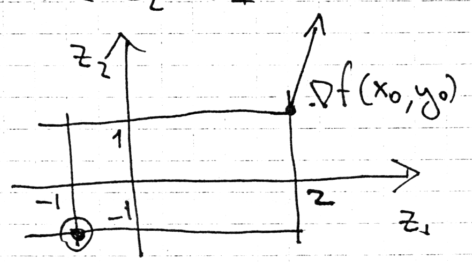
~~So~~ We can solve the two-dim. subproblem

$$\min \begin{pmatrix} 4 \\ 2 \end{pmatrix}^t \left(z - \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right)$$

$$\text{s.t.} \quad -1 \leq z_1 \leq 2$$

$$-1 \leq z_2 \leq 1$$

graphically:



$$z^* = (-1, -1)$$

Let's solve the line search problem:

$$\min_{0 \leq \alpha \leq 1} f(x(\alpha), y(\alpha))$$

where $x(\alpha) = (1-\alpha) x_0 + \alpha z_1^* = 2(1-\alpha) - \alpha = 2 - 3\alpha$

$y(\alpha) = (1-\alpha) y_0 + \alpha z_2^* = 1 \cdot (1-\alpha) - \alpha = 1 - 2\alpha$

$$f(x(\alpha), y(\alpha)) = (2-3\alpha)^2 + (1-2\alpha)^2 = 4 - 12\alpha + 9\alpha^2 + 1 - 4\alpha + 4\alpha^2 =$$
$$= 13\alpha^2 - 16\alpha + 5$$

The min is achieved at $\alpha^* = \frac{16}{26} = \frac{8}{13}$ (e.g. $f_\alpha'(x(\alpha), y(\alpha)) =$

$\Rightarrow$ can calculate $(x(\alpha^*), y(\alpha^*)) = \left(\frac{2}{13}, -\frac{3}{13}\right)$.

This is, of course, not the optimal solution $(x^*, y^*) = (0, 0)$
(objective f-n is positive $\forall (x, y) \neq (x^*, y^*)$, while
$f^* = f(x^*, y^*) = 0$).

Question 4.

b)

The function f is convex, as well as the set $X = \{ x \mid Ax \leq b \}$. Therefore, $\forall x \in X$ we have:

$$f(x_k) + \nabla f^t(x_k)(x - x_k) \leq f(x).$$

Taking the min over $x \in X$ of the both sides yelds:

$$f(x_k) + \min_{x \in X} \nabla f^t(x_k)(x - x_k) \leq \min_{x \in X} f(x),$$

which is the same as

$$f(x_k) + L_k^* \leq f^*, \text{or} \quad f(x_k) - f^* \leq -L_k^*$$

■

5\. a) $x^* = (0,0)^T$.

b) The interior penalty function is

$$x_1 + x_2 - \sigma \ln(-x_1^2 + x_2) - \sigma \ln(x_1).$$

In $\mathbb{R}^2_{++}$, this function is in C^2. Differentiating yields that

$$1 + \frac{\sigma \cdot 2x_1}{-x_1^2 + x_2} - \frac{\sigma}{x_1} = 0 \quad \text{and}$$

$$1 - \frac{\sigma}{-x_1^2 + x_2} = 0,$$

that is, $$x_1(\sigma) = (-1 + \sqrt{1+8\sigma})/4,$$

$$x_2(\sigma) = \frac{(-1 + \sqrt{1+8\sigma})^2}{16} + \sigma.$$

As $\sigma \downarrow 0$, $x_1(\sigma) \downarrow 0$ and $x_2(\sigma) \downarrow 0$ so

$(x_1, x_2) \to (0,0)$.

6a) The semi-assignment problem is possible to solve as n simple problems of the form

$$\text{minimize } \sum_i c_{ij} x_{ij}$$

$$\text{subject to } \sum_i x_{ij} = 1, \qquad (j = 1, \dots, n)$$

$$x_{ij} \geq 0, \quad i = 1, \dots, n.$$

The reason is that the problem is linear and separable over j. The extreme points of the feasible set of the above problem are all integer; they are of the form $\mathbb{R}^n \ni \bar{x} = (0, 0, \dots, 1, 0, \dots, 0)^T$, where there is exactly one element with value 1, and the remaining elements are zero. Solving the problem using the simplex method, for example, yields an extreme point automatically, and hence an optimal problem to the binary problem where "$x_{ij} \geq 0$" is replaced by "$x_{ij} \in \{0, 1\}$".

b) (P_j) minimize $\sum_{i=1}^{n} c_{ij} x_{ij}$

subject to $\begin{cases} \sum_{i=1}^{n} x_{ij} = 1, & \mid \pi \\ x_{ij} \geq 0, \quad i = 1, \ldots, n \end{cases}$

(D_j) maximize π

subject to $\pi \leq c_{ij}, \quad i = 1, \ldots, n$

To solve (D_j), it is clear that the optimal solution is to set

$$\pi^* = \min_i \{c_{ij}\}.$$

The optimality conditions state, by complementarity, that

$$x_{ij}^* \cdot (c_{ij} - \pi^*) = 0, \quad i = 1, \ldots, n.$$

Taking any i for which $c_{ij} = \pi^*$ (at least one must exist by the definition of π^*), we then set $x_{ij}^* = 1$ for this index, say, i^*, and $x_{ij} = 0$ for all $i \neq i^*$.

This is a primal-dual solution that satisfies the primal constraints ($\sum x_{ij}^* = 1$, $x_{ij}^* \geq 0 \; \forall i$), dual constraints ($\pi^* \leq c_{ij}, \forall i$) and complementarity. It is therefore optimal. The procedure described is exactly the one described here. We are done.

Question 7.

Let f^* be ~~the~~ optimal value of the problem (P); let further x_1^*, x_2^* be two optimal solutions to (P).

Since f is convex

$$f^* = f(x_1^*) \geq f(x_2^*) + \nabla f^t(x_2^*)(x_1^* - x_2^*) =$$
$$= f^* + \nabla f^t(x_2^*)(x_1^* - x_2^*)$$

$$\Rightarrow \nabla f^t(x_2^*)(x_1^* - x_2^*) \leq 0$$

On the other hand, since x_2^* is an optimal solution, and $x_1^* \in X \Rightarrow$

$$\nabla f^t(x_2^*)(x_1^* - x_2^*) \geq 0 \quad \text{[optimality conditions]}$$

Therefore, $\nabla f^t(x_2^*)(x_1^* - x_2^*) = 0.$ $\qquad (*)$

Now, $\forall y \in \mathbb{R}^n$:

$$f(y) \geq f(x_2^*) + \nabla f^t(x_2^*)(y - x_2^*) \quad \text{[by convexity]}$$
$$= f^* + \nabla f^t(x_2^*)(y - x_1^*) + \nabla f^t(x_2^*)(x_1^* - x_2^*)$$
$$= f(x_1^*) + \nabla f^t(x_2^*)(y - x_1^*). \quad \Big| \text{ because } f(x_1^*) = f^* \text{ and } (*)$$

Therefore, $\nabla f(x_2^*) \in \partial f(x_1^*)$!

But $\partial f(x_1^*) = \{\nabla f(x_1^*)\} \Rightarrow$

$\nabla f(x_1^*) = \nabla f(x_2^*)$ for arbitrary optimal solutions x_1^*, x_2^* ∎