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Solution to question 1: (a) By introducing a slack variable x_4 and two artificial variables x_5 and x_6 we get the Phase I problem

$$\begin{array}{llllllll} \text{minimize } w = & & & & & +x_5 & +x_6 & \\ \text{subject to} & 2x_1 & & +x_3 & -x_4 & +x_5 & & = 3, \\ & 2x_1 & +2x_2 & +x_3 & & & +x_6 & = 5, \\ & x_1, & x_2, & x_3, & x_4, & x_5, & x_6 & \geq 0. \end{array}$$

Let $x_B = [x_5, x_6]$ and $x_N = [x_1, x_2, x_3, x_4]$ be the initial basic and nonbasic vector respectively. The reduced costs of the nonbasic variables then become

$$c_N^T - c_B^T B^{-1} N = [-4, -2, -2, 1],$$

and thus x_1 is the entering variable. Further, we have

$$B^{-1}b = [3, 5]^T,$$
$$B^{-1}N_1 = [2, 2]^T,$$

which gives

$$\underset{j,(B^{-1}N_1)_j>0}{\arg\min} \frac{(B^{-1}b)_j}{(B^{-1}N_1)_j} = 1,$$

so x_5 is the leaving variable. The new basic and nonbasic vectors are $x_B = [x_1, x_6]$ and $x_N = [x_5, x_2, x_3, x_4]$, and the reduced costs of the nonbasic variables become

$$c_N^T - c_B^T B^{-1} N = [2, -2, 0, -1],$$

so x_2 is the entering variable, and

$$B^{-1}b = [1.5, 2]^T,$$
$$B^{-1}N_2 = [0, 2]^T,$$

which gives

$$\underset{j,(B^{-1}N_2)_j>0}{\arg\min} \frac{(B^{-1}b)_j}{(B^{-1}N_2)_j} = 2,$$

and thus x_6 is the leaving variable. The new basic and nonbasic vectors become $x_B = [x_1, x_2]$ and $x_N = [x_5, x_6, x_3, x_4]$, and the reduced costs of the nonbasic variables are

$$c_N^T - c_B^T B^{-1} N = [1, 1, 0, 0],$$

so $x_B = [x_1, x_2]$ is an optimal basis to the Phase I problem, and $w^* = 0$. This means that $x_B = [x_1, x_2]$ gives a basic feasible solution to the Phase II problem, that is,

$$\begin{array}{llllll} \text{minimize } z = & 3x_1 & +x_2 & +x_3 & & \\ \text{subject to} & 2x_1 & & +x_3 & -x_4 & = 3, \\ & 2x_1 & +2x_2 & +x_3 & & = 5, \\ & x_1, & x_2, & x_3, & x_4 & \geq 0. \end{array}$$

If $x_B = [x_1, x_2]$ and $x_N = [x_3, x_4]$ we get the reduced costs

$$c_N^T - c_B^T B^{-1} N = [-0.5, 1],$$

which means that x_3 is the entering variable, and

$$B^{-1}b = [1.5, 1]^T,$$
$$B^{-1}N_1 = [0.5, 0]^T,$$

which gives

$$\underset{j,\,(B^{-1}N_1)_j>0}{\arg\min} \frac{(B^{-1}b)_j}{(B^{-1}N_1)_j} = 1,$$

so x_1 is the leaving variable. We get $x_B = [x_3, x_2]$ and $x_N = [x_1, x_4]$, and the reduced costs become

$$c_N^T - c_B^T B^{-1} N = [1, 0.5],$$

so $x_B = [x_3, x_2]$ is an optimal basis, and since

$$B^{-1}b = [3, 1]^T$$

an optimal solution is given by

$$x^* = [x_1, x_2, x_3] = [0, 1, 3],$$

and $z^* = 4$.

(b) The reduced cost is

$$c_N^T - c_B^T B^{-1} N = [3, 0] - [1, c] \begin{pmatrix} 1 & 0 \\ -0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & 0 \end{pmatrix} = [1, 1 - 0.5c]$$

which gives $c \leq 2$. $\square$

Solution to problem 2: The dual to the given LP-problem is

$$\begin{array}{lrrl} \text{maximize } w = & -cx & +b^T y & \\ \text{subject to} & Ax & & \geq b, \\ & & -A^T y & \geq -c^T, \\ & x, & y & \geq 0, \end{array}$$

which means that the primal problem and the dual problem has the same set of feasible solutions. Thus, the primal problem is feasible if and only if the dual problem is feasible and weak duality gives that the problem cannot be unbounded. Suppose that $z^* > 0$. Strong duality gives that there exists a dual solution such that $w^* = z^*$. But this dual solution is feasible to the primal problem and gives lower value of z than z^*, which contradicts the optimality of z^*. Now suppose that $z^* < 0$. The primal optimal solution is feasible to the dual problem and gives a dual objective value that is higher than z^*, which contradicts weak duality. □

3 solution

In order to formulate the problem, we introduce the following notation:
Parameters:

- m Number of customer areas.
- n Number of possible store locations.
- P_i Set of locations in area i's primary region
- S_i Set of locations in area i's secondary region
- c_i Potential customers in area i
- s_j Maximum capacity of store j.
- r_j Annual cost of ruining a store at location j.
- q Annual income per customer

Variables:

- y_j Binary variable indicating if a store is opened at location i
- x_{ij} Customers from area i shopping at location j.

$$\max \quad \sum_{i=1}^{m} \sum_{j \in P_i \cup S_i} x_{ij} q - \sum_{j=1}^{n} r_j y_j \tag{1.1}$$

$$\text{s.t} \quad \sum_{j \in P_i} x_{ij} + \sum_{j \in S_i} 2x_{ij} \leq c_j, \quad i = 1, \ldots, M \tag{1.2}$$

$$\sum_{i=1}^{m} x_{ij} \leq y_j s_j, \quad j = 1, \ldots, n \tag{1.3}$$

Here, (1.1) measures the income from served customers reduced with the running costs. Equation (1.2) guarantees that more customers does not come from an area than what is actually there. Equation (1.3) makes sure that people only shop in open stores.

4 a) A function $f: \mathbb{R}^n \mapsto \mathbb{R}$ is convex on $\mathbb{R}^n$ if for every $x, y \in \mathbb{R}^n$ and $\lambda \in (0,1)$,

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y).$$

A function $f: \mathbb{R}^n \mapsto \mathbb{R}$ is strictly convex on $\mathbb{R}^n$ if for every $x, y \in \mathbb{R}^n$ with $x \neq y$ and $\lambda \in (0,1)$,

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y).$$

Every affine function, $f(x) = c^T x - q$, $c \in \mathbb{R}^n$, $q \in \mathbb{R}$, is convex on $\mathbb{R}^n$, but <u>not</u> strictly convex.

The function $f(x) = x^4$ is strictly convex and differentiable on $\mathbb{R}$, but since $f''(0) = 0$, its Hessian (here, second derivative) is not positive everywhere.

b) Given the system $Ax = b$; $x \ge 0^n$ where $A \sim m \times n$, $b \in \mathbb{R}^m$, $n > m$ and A has full row rank, a basic feasible solution is a solution to $B x_B = b$, where $x_B \ge 0^m$, B being an invertible square $m \times m$ matrix formed as follows: $A = (B, N)$; $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$, that is, as a collection of m column vectors of A. x_B is degenerate if x_B contains a zero element.

5. a) With λ being the Lagrangemultiplier, the KKT conditions are:

$$2c_j x_j - \lambda = 0, \quad j = 1, \dots, n \qquad (1)$$

$$\sum_j x_j = b \qquad (2)$$

From (1) follows that $x_j = \frac{\lambda}{2c_j}$, $j = 1, \dots, n$,

so by (2), $\frac{\lambda}{2}\sum_j \frac{1}{c_j} = b$, and so

$\lambda^* = \frac{2b}{\sum_j 1/c_j}$ must hold. Hence,

$x_j^* = b/(c_j \cdot \sum_k 1/c_k)$ is a KKT point. It is easy to check that f is strictly convex and that it is a convex problem, so x^* is unique.

$\therefore$ $x_j^* = p_j(c, b)$, where

$$p_j(c, b) = \frac{b}{c_j \cdot \sum_k \frac{1}{c_k}}$$

p_j is increasing in b. (Natural, as b can be considered as a demand that must be fulfilled, at least cost.)

p_j is decreasing in c_j. (Natural, because it makes x_j less favorable compared to the other variables.)

b) Consider the function

$$\rho_j(c,b) = \frac{b}{c_j \cdot \sum_k \frac{1}{c_k}}, \quad j=1,\dots,n,$$

over $b>0$, $c>0^n$. With $c>0^n$ fixed, it is clearly a linear function of b. It is therefore differentiable, with

$$\frac{\partial \rho_j(c,b)}{\partial b} = \frac{1}{c_j \cdot \sum_k \frac{1}{c_k}}.$$

$$\therefore \quad \nabla_b x_j^* = \rho_j'(b) = \frac{\partial \rho_j(c,b)}{\partial b} = \frac{1}{c_j \cdot \sum_k \frac{1}{c_k}}.$$

The derivative is positive, showing (as discussed in a)) that the variables x_j <u>all</u> increase at the optimum solution when the value of b increases.

Question 6.

a) $\nabla\left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right) = X$, therefore

$$X^{t+1} = X^t - sX^t = (1-s)^{t+1}X^0$$

The latter converges either

$\forall s \geq 0$, if $X^0 = 0$ [trivial case]

or

$\forall x^0 \in \mathbb{R}^n$, if $|1-s| < 1$, i.e. $0 < s < 2$.

b) Trivially, $\|\nabla f(x) - \nabla f(y)\| \leq 1 \cdot \|x-y\|$, i.e. $L = 1$ (Lipschitz constant).

Therefore, from the general theory it follows that algorithm converges

$\forall\ 0 < s < 2/1 = 2$,

which is the same as we obtained in a) [excluding trivial case].

Question 7.

a) Let $x \in \text{hull}(S_1 \cap S_2)$, i.e.
$x = \lambda y + (1-\lambda) z$ for some $0 \leq \lambda \leq 1$,
$y \in S_1 \cap S_2$, $z \in S_1 \cap S_2$.

Then, $x \in \text{hull}(S_1)$ (since $y, z \in S_1$)
& $x \in \text{hull}(S_2)$ (since $y, z \in S_2$),
finishing the proof.

b) Let $S_1 = \{-1, 1\} \subset \mathbb{R}$
$S_2 = \{0\} \subset \mathbb{R}$
Then, $\text{hull}(S_1) = [-1, 1]$
$\text{hull}(S_2) = \{0\}$ so that
$\text{hull}(S_1) \cap \text{hull}(S_2) = [-1, 1]$, while
$\text{hull}(S_1 \cap S_2) = \text{hull}(\emptyset) = \emptyset$.