Chalmers/GU
Mathematics

EXAM SOLUTION

TMA947/MAN280
APPLIED OPTIMIZATION

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Question 1

(the Simplex method)

(2p) a) After changing sign of the second inequality and adding two slack variables s_1 and s_2, a BFS cannot be found directly. We create the phase I problem through an added artificial variable a_1 in the second linear constraint; the value of a_1 is to be minimized.

We use the BFS based on the variable pair (s_1, a_1) as the starting BFS for the phase I problem. In the first iteration of the Simplex method x_1 is the only variable with a negative reduced cost; hence x_1 is picked as the incoming variable. The minimum ratio test shows that s_1 should leave the basis. In the next iteration the reduced cost for varialbe x_3 is negative, and x_3 is picked as the incoming variable. The minimum ratio test shows that a_1 should leave the basis. We have found an optimal basis, $x_B = (x_1, x_3)^T$, to the phase I problem. We proceed to phase II, since the basis is feasible in the original problem.

Starting phase II with this BFS, we see that all reduced costs are positive, $\tilde{c}_N = (\alpha + 4, 2, 3)^T > 0$, and thus the BFS is optimal. $x_B = B^{-1}b = (2, 1)^T$ so $x^* = (2, 0, 3)^T$ and $z^* = c_B^T x_B = 3$.

(1p) b) For the dual problem to be unbounded, weak duality shows that the primal problem must be infeasible. Since α is in the cost vector of the primal problem, the feasibilty is not affected by α. Hence, no values of α lead to an unbounded feasible problem.

(3p)

Question 2

(necessary local and sufficient global optimality conditions)

See Proposition 4.23 and Theorem 4.24.

Question 3

(Newton's method revisited)

(2p) a) See the text book on quasi-Newton methods.

(1p) b) In order to be certain that the search direction given by the Newton subproblem is (a) defined at all and (b) is a direction of descent, the Hessian matrix must be positive definite. There are several ways in which to modify a matrix that is not positive definite such that the resulting matrix has this property.

The classic one is the Levenberg–Marquardt modification, in which one adds a diagonal matrix to the Hessian matrix such that their sum is positive definite. A second possibility is to replace Newton's method altogether with a quasi-Newton method, as explained in a). Special modifications also include the use of directions of negative curvature, in case the Hessian matrix is indefinite. See the text book for more details.

Question 4

(modelling)

Introduce the binary variables

$$x_i = \begin{cases} 1 & \text{if team } i \text{ is placed in group 1} \\ 0 & \text{otherwise} \end{cases}, \qquad i = 1, \ldots, 14$$

The objective function can then be written as

$$\min \sum_{i=1}^{13} \sum_{j=i+1}^{14} d_{ij} \left(x_i x_j + (1 - x_i)(1 - x_j) \right) + \sum_{i=1}^{14} \sum_{j=1}^{14} d_{ij} \frac{p_i x_i}{\sum_{k=1}^{14} p_k x_k} \cdot \frac{p_j (1 - x_j)}{\sum_{k=1}^{14} p_k (1 - x_k)}$$

where the first term represents the travelling within the groups and the second term the expected travelling in the final. The constraints are

$$\begin{aligned} \sum_{i=1}^{14} x_i &= 7, && (1) \\ x_1 + x_2 &= 1 && (2) \\ \sum_{i=1}^{14} x_i p_i &\leq \sum_{i=1}^{14} (1 - x_i) p_i + 0.15 \sum_{i=1}^{14} p_i, && (3) \\ \sum_{i=1}^{14} (1 - x_i) p_i &\leq \sum_{i=1}^{14} x_i p_i + 0.15 \sum_{i=1}^{14} p_i, && (4) \\ x_i &\in \{0, 1\} \quad i = 1, \ldots, 14. && (5) \end{aligned}$$

Constraint (1) makes sure that there are 7 teams in each group, constraint (2) that the two best teams are not in the same group and the constraints (3) and

(4) that the groups are arranged so that the difference between the sum of points in the two groups are not bigger than 15% of the total points.

Another possibility (maybe better) is to introduce more binary variables, u_{ij} and v_{ij}, where

$$u_{ij} = \begin{cases} 1 & \text{if team } i \text{ and team } i \text{ are both placed in group 1} \\ 0 & \text{otherwise} \end{cases}, \quad i, j = 1, \dots, 14,$$

$$v_{ij} = \begin{cases} 1 & \text{if team } i \text{ and team } i \text{ are both placed in group 2} \\ 0 & \text{otherwise} \end{cases}, \quad i, j = 1, \dots, 14.$$

We can then add to the previous model the linear forcing constraints

$$\begin{aligned} x_i + x_j &\leq u_{ij} + 1 \quad i, j = 1, \dots, 14, &(6) \\ x_i + x_j &\geq 1 - v_{ij} \quad i, j = 1, \dots, 14, &(7) \end{aligned}$$

the binary constraints

$$\begin{aligned} u_{ij} &\in \{0, 1\} \quad i, j = 1, \dots, 14, &(8) \\ v_{ij} &\in \{0, 1\} \quad i, j = 1, \dots, 14, &(9) \end{aligned}$$

and replace the first term in the previous objective function with the simpler linear term

$$\sum_{i=1}^{13} \sum_{j=i+1}^{14} d_{ij} \left(u_{ij} + v_{ij} \right).$$

Question 5

(interior penalty methods)

(1p) a) All functions involved are in C^1. The conditions on the penalty function are fulfilled, since $\phi'(s) = 1/s^2 \geq 0$ for all $s < 0$. Further, LICQ holds everywhere. The answer is yes.

(2p) b) With the given data, it is clear that the only constraint is (almost) fulfilled with equality: $(\boldsymbol{x}_6)_1^2 - (\boldsymbol{x}_6)_2 \approx -0.005422 \approx 0$. We set up the KKT conditions to see whether it is fulfilled approximately. Indeed, we have the following corresponding to the system $\nabla f(\boldsymbol{x}_6) + \hat{\mu}_6 \nabla g(\boldsymbol{x}_6) = \mathbf{0}^2$:

$$\begin{pmatrix} -6.4094265 \\ 3.39524 \end{pmatrix} + 3.385 \begin{pmatrix} 1.88778 \\ -1 \end{pmatrix} \approx \begin{pmatrix} -0.01929 \\ 0.01024 \end{pmatrix},$$

and the right-hand side can be considered near-zero. Since $\hat{\mu}_6 \geq 0$ we approximately fulfill the KKT conditions.

For the last part, we establish that the problem is convex. The feasible set clearly is convex, since g is a convex function and the constraint is on the "$\leq$"-form. The Hessian matrix of f is

$$\begin{pmatrix} 12(x_1-2)^2+2 & -4 \\ -4 & 8 \end{pmatrix},$$

which is positive semidefinite everywhere (in fact, positive definite outside of the region defined by $x_1 = 2$); hence, f is convex on $\mathbb{R}^2$. We conclude that our problem is convex, and hence the KKT conditions imply global optimality. The vector $\boldsymbol{x}_6$ therefore is an approximate global optimal solution to our problem.

Question 6

(linear programming)

(1p) a) By complementarity slackness (Theorem 10.12),

$$\boldsymbol{x}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{y}-\boldsymbol{c}) = 0 \Leftrightarrow \left\{ \begin{array}{l} x_1(y_1+4y_2-2) = x_1 \cdot 0 = 0, \\ x_2(-y_1+2y_2+2) = x_2 \cdot 2 = 0 \Rightarrow x_2 = 0, \\ x_3(2y_1-y_2-1) = x_3 \cdot 0 = 0. \end{array} \right\}$$

Further, it follows that

$$\boldsymbol{y}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{x}-\boldsymbol{b}) = 0 \Leftrightarrow \left\{ \begin{array}{l} x_1+2x_3 = 1, \\ 4x_1-x_3 = 2, \\ x_2 = 0 \end{array} \right\} \Leftrightarrow \{x_1 = 5/9,\ x_2 = 0,\ x_3 = 2/9\}.$$

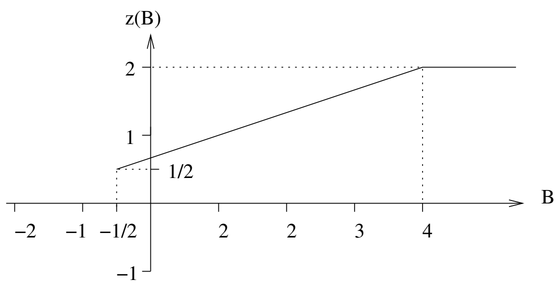
Since $\boldsymbol{x} \geq \boldsymbol{0}$ and $\boldsymbol{y} \geq \boldsymbol{0}$ it follows that $\boldsymbol{y} = (2/3, 1/3)$ is an optimal solution to the LP dual problem.

(2p) b) For $\beta = 2$ the optimal basis is $\boldsymbol{x}_B = (x_1, x_3)^{\mathrm{T}}$. This holds for those values of β such that $\boldsymbol{x}_B$ is feasible and optimal. Here, $\boldsymbol{B} = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$, so that $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} = 1/9 \cdot \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ \beta \end{pmatrix} = 1/9 \cdot \begin{pmatrix} 1+2\beta \\ 4-\beta \end{pmatrix} \geq 0 \Leftrightarrow$ $-1/2 \leq \beta \leq 4$. Optimality follows from $\boldsymbol{c}_N^{\mathrm{T}} - \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{N} = (-2, 0, 0) -$

$(2,1)\cdot 1/9\cdot\begin{pmatrix}1 & 2\\ 4 & -1\end{pmatrix}\begin{pmatrix}-1 & 1 & 0\\ 2 & 0 & 1\end{pmatrix} = (-2,-2/3,-1/3) \le 0$. Within the interval $-1/2 \le \beta \le 4$, $z(\beta) = (2+\beta)/3$.

For $\beta > 4$, x_3 becomes negative $\Rightarrow$ x_3 is not in the optimal basis for $\beta > 4$. Entering variable (according to the criterion in the dual simplex method) is x_2. The next basis is $\boldsymbol{x}_B = (x_1, x_2) = \boldsymbol{B}^{-1}\boldsymbol{b} = 1/6\cdot\begin{pmatrix}2 & 1\\ -4 & 1\end{pmatrix}\begin{pmatrix}1\\ \beta\end{pmatrix} = 1/6\cdot\begin{pmatrix}2+\beta\\ \beta-4\end{pmatrix}$, which is optimal, since $\boldsymbol{c}_N^{\mathrm{T}} - \boldsymbol{c}_B^{\mathrm{T}}\boldsymbol{B}^{-1}\boldsymbol{N} = (1,0,0)-(2,-2)\cdot 1/6\cdot\begin{pmatrix}2 & 1\\ -4 & 1\end{pmatrix}\begin{pmatrix}2 & 1 & 0\\ -1 & 0 & 1\end{pmatrix} = (-3,-2,0) \le 0$. Feasibility holds for $\boldsymbol{x}_B \ge 0 \Leftrightarrow \beta \ge 4$. Hence, for $\beta \ge 4$, $z(\beta) = 2$.

The function $z(\beta)$ is piecewise linear and concave on the halfline $\beta \ge -1/2$.



Question 7

(Lagrangian duality)

(1p) a) The Lagrangian subproblem is to, for any $\mu \ge 0$,

$$\begin{aligned}\text{minimize } & x_2 - \mu(2x_1 + 3x_2 - 6),\\ \text{subject to } & x_1 \in [0, 3/2],\\ & x_2 \ge 0.\end{aligned}$$

This problem has the following solution sets for varying values of μ: for $\mu \in [0, 1/3)$, $\boldsymbol{x}(\mu) = (3/2, 0)^{\mathrm{T}}$ uniquely; for $\mu = 1/3$, $x_1(\mu) = 3/2$ while $x_2(\mu) \ge 0$ arbitrarily; finally, for $\mu > 1/3$, there exists no optimal solution to the Lagrangian subproblem.

Inserting these solutions into the Lagrangian subproblem we obtain that the Lagrangian dual function has the following appearance: for $\mu \in [0, 1/3]$, $q(\mu) = 6\mu - 3\mu = 3\mu$, while for $\mu > 1/3$, $q(\mu) = -\infty$.

We can therefore state an explicit linear dual problem as follows:

$$\begin{aligned} &\text{maximize } 3\mu, \\ &\text{subject to } 0 \leq \mu \leq 1/3. \end{aligned}$$

(1p) b) $\boldsymbol{x} = (1, 2)^{\mathrm{T}} \implies z = 2$; $\boldsymbol{x} = (1, 4/3)^{\mathrm{T}} \implies z = 4/3$; $\boldsymbol{x} = (3/2, 1)^{\mathrm{T}} \implies z = 1$.

$\mu = 0 \implies q(\mu) = 0$; $\mu = 1/6 \implies q(\mu) = 1/2$; $\mu = 1/3 \implies q(\mu) = 1$.

(1p) c) $\mu^* = 1/3$. From a) the optimality conditions for the Lagrangian subproblem yields that $x_1(\mu^*) = x_1^* = 3/2$, while $x_2(\mu^*) \geq 0$. Since $\mu^* \neq 0$, we must satisfy the Lagrangian relaxed constraint with equality; this yields the condition that $3 + 3x_2 = 6$, hence $x_2 = 1$. We verify that $z^* = q^* = 1$.