Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MMG620
OPTIMIZATION, BASIC COURSE

Date:	10–12–13
Time:	House V, morning
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Emil Gustavsson (0703-088304)
Result announced:	11–01–05 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{aligned} \text{maximize} \quad & 3x_1 + x_2, \\ \text{subject to} \quad & 3x_1 + 2x_2 \geq 1, \\ & 2x_1 + x_2 \leq 2, \\ & x_1 \geq 0, \\ & x_2 \in \mathbb{R}. \end{aligned}$$

(2p) a) Solve this problem using phase I and phase II of the simplex method.

Aid: Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(1p) b) Is the solution obtained unique? Motivate your answer!

Question 2

(optimality conditions)

Suppose that for $j = 1, \ldots, n$ the functions $f_j : \mathbb{R} \to \mathbb{R}$ are convex and differentiable. Let $b > 0$. Our problem, called the *resource allocation problem*, has the following general statement:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \sum_{j=1}^{n} f_j(x_j), && \text{(1a)} \\ \text{subject to} \quad & \sum_{j=1}^{n} x_j = b, && \text{(1b)} \\ & x_j \geq 0, \qquad j = 1, \ldots, n. && \text{(1c)} \end{aligned}$$

While this problem is exceptionally simple it has many applications, for example in portfolio optimization, production economics, and stratified sampling.

(2p) a) Introduce any necessary multipliers, and describe the necessary Karush–Kuhn–Tucker conditions for a vector $\boldsymbol{x}^*$ to be a local optimum in the problem (1). Are these conditions also sufficient for the global optimality of a KKT-point $\boldsymbol{x}^*$?

(1p) b) Based on the result in a), establish the following result on the characterization of optimal solutions to the problem (1), known as Gibbs' Lemma: *Suppose that $\boldsymbol{x}^*$ solves the problem* (1)*. Then, there exists (at least one) $\lambda^* \in \mathbb{R}$ such that*

$$f_j'(x_j^*) \begin{cases} = \lambda^*, & \text{if } x_j^* > 0, \\ \geq \lambda^*, & \text{if } x_j^* = 0, \end{cases} \qquad j = 1, \ldots, n, \tag{2}$$

holds.

Question 3

(modeling)

A group of n people has decided to arrange a party for New Year's Eve. Each member of the group has purchased things to the party for d_i SEK, $i \in \{1, \ldots, n\}$. Your assignment is to decide how money should be transferred between the members such that all members will have paid equally much.

(1p) a) Introduce the necessary variables and formulate a linear programming (LP) model which minimizes the total amount of transfered cash between the members.

(2p) b) In the solution to the first model, one of the participants is supposed to give money to six other members; this is fairly impractical. Introduce additional integer variables and extend your model to a linear mixed integer programming model (i.e., the model should be linear if the integrality constraints are relaxed) such that *each member only needs to give money to at most one other member.*

(3p) Question 4

(the Frank–Wolfe method)

Consider the problem to

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \;\; & f(\boldsymbol{x}) := x_1^2 + x_2^2 + x_1 x_2 - 3x_1 - 6x_2, \\ \text{subject to} \;\; & \begin{cases} x_1 + x_2 \le 4, \\ -2x_1 + x_2 \le 0, \\ 0 \le x_2 \le 2. \end{cases} \end{aligned}$$

Start at the point $\boldsymbol{x}_0 = (0,0)^{\mathrm{T}}$ and perform two iterations of the Frank–Wolfe method. (Recall that the Frank-Wolfe method starts at some feasible point. Given an iteration k and feasible iterate $\boldsymbol{x}_k$ it produces a feasible search direction $\boldsymbol{p}_k$ through the minimization of the fist-order Taylor expansson of f at $\boldsymbol{x}_k$. The next iterate is found through an exact line search in f along the search direction, such that the resulting vector is also feasible.)

Give the upper and lower bounds of the optimal objective function value that the algorithm generates in each iteration, and give a theoretical motivation for them. If an optimum is found, motivate why it is an optimum.

Question 5

(Lagrangian duality)

Consider the problem to find the Euclidean projection of the origin in $\mathbb{R}^2$ on the polyhedral set defined by the three linear inequalities $x_1 \le 4$, $x_2 \le 4$, and $x_1 + x_2 \ge 4$.

(1p) a) State this projection problem as a convex quadratic optimization problem.

(1p) b) By Lagrangian relaxing the constraint that $x_1 + x_2 \ge 4$ must hold, formulate the corresponding Lagrangian dual problem *explicitly*. Establish that the Lagrangian dual problem is that of maximizing a concave function.

(1p) c) Solve this Lagrangian dual problem. Utilize the primal–dual relationships between the primal and the dual problem to establish the solution to the original problem. Confirm your answer graphically.

(3p)
Question 6

(optimality conditions)

Farkas' Lemma can be stated as follows:

Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{b}$ an $m \times 1$ vector. Then exactly one of the systems

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b}, \\ \boldsymbol{x} &\geq \boldsymbol{0}^n, \end{aligned} \tag{I}$$

and

$$\begin{aligned} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} &\leq \boldsymbol{0}^n, \\ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} &> 0, \end{aligned} \tag{II}$$

has a feasible solution, and the other system is inconsistent.

Prove Farkas' Lemma.

Question 7

(short questions)

Answer the following three short questions. You *must* motivate your answers in order to receive any points.

(1p) a) Consider the following problem

$$\underset{x \in X}{\text{minimize}}\, f(\boldsymbol{x}),$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $X \subset \mathbb{R}^n$ is a convex and bounded set. Assume further that for some $M \in \mathbb{R}$ $f(\boldsymbol{x}) \geq M$ for all $\boldsymbol{x} \in X$ holds. Does an optimal solution always exist to this problem? If not, give a counter-example!

(1p) b) Assume that the objective function $f : \mathbb{R}^2 \to \mathbb{R}$ is a convex function. Define the constraint functions as

$$g_1(x, y) := \begin{cases} (x+1)^2 + (y+1)^2 - 1, & \text{for } x < -1, \\ (y+1)^2 - 1, & \text{for } -1 \leq x \leq 1, \\ (x-1)^2 + (y+1)^2 - 1, & \text{for } 1 < x, \end{cases}$$

and

$$g_2(x, y) := -y.$$

Consider the following problem:

$$\begin{aligned} \text{minimize} \quad & f(x), \\ \text{subject to} \quad & g_1(x, y) \leq 0, \\ & g_2(x, y) \leq 0. \end{aligned}$$

Answer the following two questions about the problem described above. If a feasible point x satisifies the KKT conditions, does it then imply that the point is optimal? If a feasible point x is optimal, does it then imply that it satisfies the KKT conditions?

(1p) c) Consider the following set

$$X = \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n (x_i - a_i)^2 \leq b,\ \exp\left(\sqrt{\sum_{i=1}^n (x_i - c_i)^2} \right) \geq d \right\}.$$

Assume that for the constants $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{c} \in \mathbb{R}^n$, $0 < d \in \mathbb{R}$ and $0 < b \in \mathbb{R}$ the following inequality holds

$$\sqrt{\sum_{i=1}^n (a_i - c_i)^2} \geq \sqrt{b} + \ln d.$$

Is the set X convex?

Good luck!