Lecture 8: Linear programming models

Michael Patriksson

2010-11-17

Duality and optimality

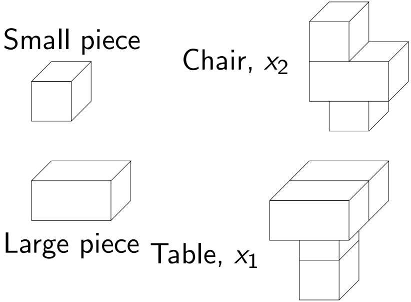
- LP: linear objective, linear constraints
- LP problems can be given a "standard form"
- LP problems are convex problems with a CQ fulfilled (linear constraints $\Longrightarrow$ Abadie)
- Strong duality holds; Lagrangian dual same as LP dual
- KKT necessary and sufficient!
- Simplex method: always satisfies complementarity; always primal feasibility after finding the first feasible solution; searches for a dual feasible point

Basic method and its foundations

- Know that if there exists an optimal solution, at least one of them is an extreme point (Thm. 8.10)
- Search only among extreme points
- Extreme points can be described in algebraic terms (Thm. 3.17). Find such a point
- Generate a descent direction; line search leads to the boundary! Choose direction so that the boundary point is an extreme point
- $\Longrightarrow$ Move to a neighbouring extreme point such that the objective value improves—the Simplex method!
- Convergence *finite*

A DUPLO game, I

- A manufacturer produces two pieces of furniture: tables and chairs
- The production of furniture requires two different pieces of raw-material, large and small pieces
- One table is assembled from two pieces of each; one chair is assembled from one of the larger pieces and two of the smaller pieces



A DUPLO game, II

- Data: 6 large and 8 small pieces available. Selling a table gives 1600 SEK, a chair 1000 SEK
- Not trivial to choose an optimal production plan
- What is the problem and how do we solve it?
- Solution by:
  1. the DUPLO game;
  2. graphically;
  3. the Simplex method

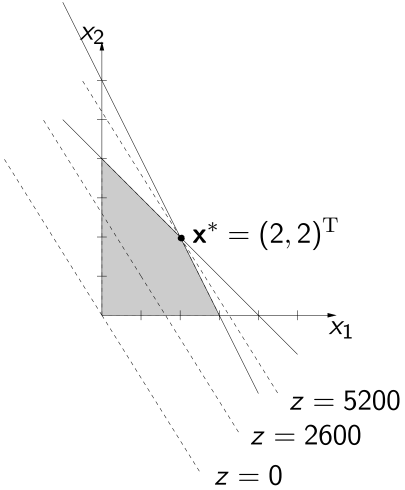
A DUPLO game, III

- Mathematical model: let x_1 and x_2 denote the number of tables and chairs produced, respectively
- The following model describes the optimization problem to find the most profitable production plan, given the resources available:

$$
\begin{array}{llrrl}
\text{maximize} & z = & 1600x_1 & +1000x_2, & \\
\text{subject to} & & 2x_1 & +x_2 & \leq 6, \\
& & 2x_1 & +2x_2 & \leq 8, \\
& & x_1, & x_2 & \geq 0
\end{array}
$$

A DUPLO game, IV

- A graphical representation of the feasible region (polyhedron) and a process to find the best solution:



Further topics

- Sensitivity analysis: What happens with z^*, $\mathbf{x}^*$ if ...?
- A dual problem: A manufacturer (Billy) produce book shelves with same raw material. Billy wish to expand their production; interested in acquiring our resources
- Two questions (with identical answers): (1) what is the lowest bid (price) for the total capacity at which we are willing to sell?; (2) what is the highest bid (price) that Billy are prepared to offer for the resources? The answer is a measure of the wealth of the company in terms of their resources (the shadow price)

A dual problem, I

- To study the problem, we introduce the variables

$$
\begin{aligned}
y_1 &= \text{the price which Billy offers for each large piece,} \\
y_2 &= \text{the price which Billy offers for each small piece,} \\
w &= \text{the total bid which Billy offers}
\end{aligned}
$$

- Example: Net income for a table is 1600 SEK; need to get at least price bid $\mathbf{y}$ such that $2y_1 + 2y_2 \geq 1600$
- Mathematical model:

$$
\begin{array}{llrrl}
\text{minimize} & w = & 6y_1 & +8y_2 & \\
\text{subject to} & & 2y_1 & +2y_2 & \geq 1600, \\
& & y_1 & +2y_2 & \geq 1000, \\
& & y_1, & y_2 & \geq 0
\end{array}
$$

- Why the sign? $\mathbf{y}$ is a price!

A dual problem, II

- Optimal solution: $\mathbf{y}^* = (600, 200)^{\mathrm{T}}$. The bid is $w^* = 5200$ SEK

- Remarks:
  1. $z^* = w^*$! (Strong duality!) Our total income is the same as the value of our resources
  2. The price for each piece equals its *shadow price*! Why?

Geometric $\iff$ Algebraic connections, I

- Must have equality constraints. Why? Inequalities cannot be manipulated while keeping the same solution set! Equalities can!

- Counter-example: add one of the inequalities to the other:

$$P = \{ \mathbf{x} \in \mathbb{R}^2_+ \mid x_1 + x_2 \geq 1; \quad 2x_1 + x_2 \leq 2 \}$$

- Good to know: Every polyhedron P can be described in the form

$$P = \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}; \quad \mathbf{x} \geq \mathbf{0}^n \}$$

- We call this the *standard form*

Geometric $\Longleftrightarrow$ Algebraic connections, II

- Slack variables: $(\mathbf{x} \in \mathbb{R}^n,\ \mathbf{b} \in \mathbb{R}^m,\ \mathbf{A} \in \mathbb{R}^{m \times n})$

$$\begin{aligned} \mathbf{A}\mathbf{x} &\leq \mathbf{b}, \\ \mathbf{x} &\geq \mathbf{0}^n \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \mathbf{A}\mathbf{x} + \mathbf{I}^m\mathbf{s} &= \mathbf{b}, \\ \mathbf{x} &\geq \mathbf{0}^n, \\ \mathbf{s} &\geq \mathbf{0}^m \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} [\mathbf{A}\ \mathbf{I}^m]\mathbf{v} &= \mathbf{b}, \\ \mathbf{v} &\geq \mathbf{0}^{n+m} \end{aligned}$$

- We assume even that $\mathbf{b} \geq \mathbf{0}^m$; otherwise, multiply necessary rows by -1

Geometric $\iff$ Algebraic connections, III

- Idea: We describe an extreme point through this characterization of the feasible set; we then prove that moving between "adjacent" extreme points is simple
- *Basic feasible solution* (Algebra) $\iff$ *Extreme point* (Geometry)
- Note: $\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} = \mathbf{b} \implies$ Linear algebra
- $\mathbf{x} \geq \mathbf{0}^n : \mathbf{Ax} = \mathbf{b} \implies$ Polyhedra, convex analysis
- Sign restrictions? If x_j is free of sign, substitute it everywhere by

$$x_j = x_j^+ - x_j^-,$$

where $x_j^+, x_j^- \geq 0$

DUPLO example with slack variables

$$
\begin{aligned}
\text{maximize}\; z = 1600x_1 + 1000x_2 & & \\
\text{subject to} \quad 2x_1 + x_2 + s_1 &= 6 & (1)\\
2x_1 + 2x_2 + s_2 &= 8 & (2)\\
x_1, \; x_2, \; s_1, \; s_2 &\geq 0 &
\end{aligned}
$$

Basic feasible solutions (BFS), I

- Consider an LP in standard form:

$$
\begin{aligned}
\text{minimize} \quad & z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\
\text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\
& \mathbf{x} \geq \mathbf{0}^n,
\end{aligned}
$$

  $\mathbf{A} \in \mathbb{R}^{m \times n}$ with rank $\mathbf{A} = m$ (otherwise, delete rows), $n > m$, and $\mathbf{b} \in \mathbb{R}^m_+$

- A point $\tilde{\mathbf{x}}$ is a *basic solution* if
  1. $\mathbf{A}\tilde{\mathbf{x}} = \mathbf{b}$; and
  2. the columns of $\mathbf{A}$ corresponding to the non-zero components of $\tilde{\mathbf{x}}$ are linearly independent (recall that $\mathbf{A}\mathbf{x} = \sum_{j=1}^n \mathbf{a}_j x_j$, where $\mathbf{a}_j$ is column j of $\mathbf{A}$)

- A basic solution that satisfies non-negativity is called a *basic feasible solution* (BFS)

Basic feasible solutions (BFS), II

- Additional terms:
  1. A *degenerate* basis is a basis where at least one component of the basic solution is zero
  2. A *non-degenerate basic solution* is strictly positive

- Connection between a BFS and an extreme point?

- *A point* $\mathbf{x}$ *is an extreme point of the set* $\{\, \mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} = \mathbf{b};\ \mathbf{x} \geq \mathbf{0}^n \,\}$ *if and only if it is a basic feasible solution*
- Proof by the fact that the rank of $\mathbf{A}$ is full + Thm. 3.17 (algebraic char. of extreme points)

The Representation Theorem revisited

Let $P = \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b};\ \mathbf{x} \geq \mathbf{0}^n \}$ *and* $V = \{\mathbf{v}^1, \dots, \mathbf{v}^k\}$ *its set of extreme points. If and only if* P *is nonempty,* V *is nonempty (finite). Let* $C = \{ \mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{0}^m;\ \mathbf{x} \geq \mathbf{0}^n \}$ *and* $D = \{\mathbf{d}^1, \dots, \mathbf{d}^r\}$ *be the set of extreme directions of* C*. If and only if* P *is unbounded* D *is nonempty (finite)*

Every $\mathbf{x} \in P$ *is the sum of a convex combination of points in* V *and a non-negative linear combination of points in* D*:*

$$\mathbf{x} = \sum_{i=1}^{k} \alpha_i \mathbf{v}^i + \sum_{j=1}^{r} \beta_j \mathbf{d}^j,$$

where $\alpha_1, \dots, \alpha_k \geq 0$*:* $\sum_{i=1}^{k} \alpha_i = 1$*, and* $\beta_1, \dots, \beta_r \geq 0$

Existence of optimal solutions to LP

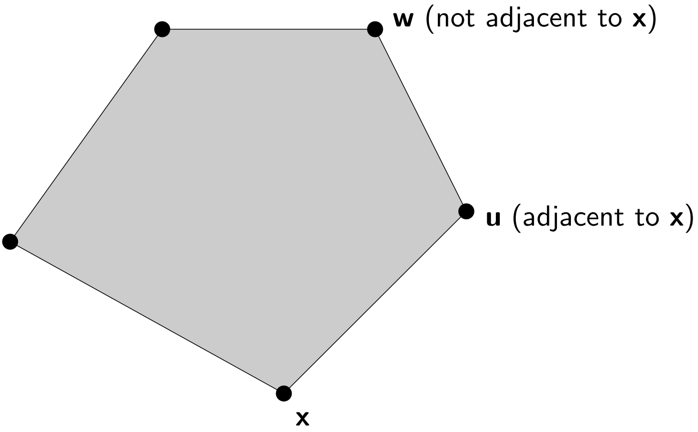
- *Let the sets P, V and D be defined as in the above theorem and consider the LP*

$$\begin{array}{ll} \text{minimize} & z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \\ \text{subject to} & \mathbf{x} \in P \end{array}$$

  This problem has a finite optimal solution if and only if P is nonempty and z is lower bounded on P, that is, if $\mathbf{c}^{\mathrm{T}}\mathbf{d}^j \geq 0$ for all $\mathbf{d}^j \in D$. If the problem has a finite optimal solution, then there exists an optimal solution among the extreme points
- *Proof.*

Adjacent extreme points, I

- Consider the following polytope:

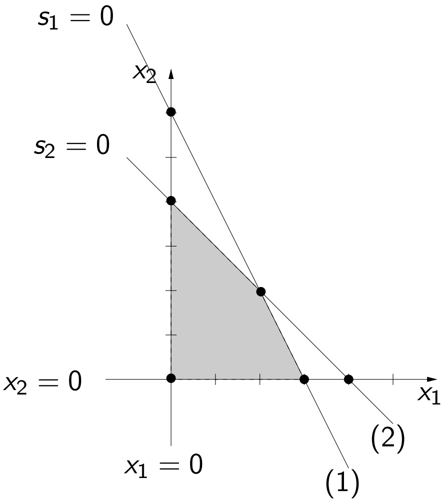


Adjacent extreme points, II

- No point on the line segment joining $\mathbf{x}$ and $\mathbf{u}$ can be written as a convex combination of any pair of points that are not on this line segment. However, this is not true for the points on the line segment between the extreme points $\mathbf{x}$ and $\mathbf{w}$. The extreme points $\mathbf{x}$ and $\mathbf{u}$ are said to be *adjacent* (while $\mathbf{x}$ and $\mathbf{w}$ are not adjacent)

- *Two extreme points are adjacent if and only if there exist corresponding BFSs whose sets of basic variables differ in exactly* one *place*

- Apply this to the DUPLO example!



$s_1 = 0$
x_2
$s_2 = 0$
$x_2 = 0$
x_1
(2)
$x_1 = 0$
(1)