Lecture 9: The Simplex Method

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An algebraic derivation of the pricing step, I

- Below is shown how, given a feasible basis (that is, a BFS), we can represent the original optimization problem as that to optimally choose the values of the non-basic variables; it provides us with a termination criterion as well as an idea for finding an improving direction if we are not at an optimal BFS:

$$
\begin{aligned}
z^* = \quad & \text{infimum } \mathbf{c}^{\mathrm{T}}\mathbf{x} & = \quad & \text{infimum } \mathbf{c}_{\mathbf{B}}^{\mathrm{T}}\mathbf{x}_{\mathbf{B}} + \mathbf{c}_{\mathbf{N}}^{\mathrm{T}}\mathbf{x}_{\mathbf{N}} \\
& \text{subject to } \mathbf{A}\mathbf{x} = \mathbf{b}, & & \text{subject to } \mathbf{B}\mathbf{x}_{\mathbf{B}} + \mathbf{N}\mathbf{x}_{\mathbf{N}} = \mathbf{b}, \\
& \mathbf{x} \geq \mathbf{0}^n & & \mathbf{x}_{\mathbf{B}} \geq \mathbf{0}^m;\ \mathbf{x}_{\mathbf{N}} \geq \mathbf{0}^{n-m}
\end{aligned}
$$

$$
\begin{aligned}
= \mathbf{c}_{\mathbf{B}}^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b} + \quad & \text{infimum } [\mathbf{c}_{\mathbf{N}}^{\mathrm{T}} - \mathbf{c}_{\mathbf{B}}^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N}]\mathbf{x}_{\mathbf{N}} \\
& \text{subject to } \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x}_{\mathbf{N}} \geq \mathbf{0}^m, \\
& \mathbf{x}_{\mathbf{N}} \geq \mathbf{0}^{n-m}
\end{aligned}
$$

- Note the consistent re-orderings:

$$
\mathbf{x} = \begin{pmatrix} \mathbf{x}_{\mathbf{B}} \\ \mathbf{x}_{\mathbf{N}} \end{pmatrix}; \quad \mathbf{A} = (\mathbf{B}, \mathbf{N}); \quad \mathbf{c} = \begin{pmatrix} \mathbf{c}_{\mathbf{B}} \\ \mathbf{c}_{\mathbf{N}} \end{pmatrix}
$$

An algebraic derivation of the pricing step, II

- $\mathbf{x_N} = \mathbf{0}^{n-m}$ feasible. Let $\tilde{\mathbf{c}}_\mathbf{N}^{\mathrm{T}} := \mathbf{c}_\mathbf{N}^{\mathrm{T}} - \mathbf{c}_\mathbf{B}^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{N}$
- If the *reduced cost* $\tilde{\mathbf{c}}_\mathbf{N} \geq \mathbf{0}^{n-m}$ then $\mathbf{x_N} = \mathbf{0}^{n-m}$ is optimal
- If $\tilde{\mathbf{c}}_\mathbf{N} \not\geq \mathbf{0}^{n-m}$ then $\exists j \in N$ with $\tilde{c}_j < 0$. Then the current point $\mathbf{x_N} = \mathbf{0}^{n-m}$ may be non-optimal
- Generate a feasible descent direction
- Choose one that leads to a neighboring extreme point
- Swap one variable in B for one in N
- Increase one variable in N from zero
- Choose $\jmath^*$ to be among $\arg\mathrm{minimum}_{j \in N}\ \tilde{c}_j$ (the "incoming" variable then is chosen as the one with the most negative reduced cost—a "steepest" descent direction along an edge)
- We have then decided on the search direction

The basis change (pivot), I

- What is this direction?
- In $\mathbf{x_N}$-space: $\mathbf{p_N} = \mathbf{e}_{\mathrm{J}^*}$ (unit vector)
- In $\mathbf{x_B}$-space: $\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\mathbf{x_N} \Longrightarrow$
  $\mathbf{p_B} = -\mathbf{B}^{-1}\mathbf{N}\mathbf{p_N} = -\mathbf{B}^{-1}\mathbf{N}_{\mathrm{J}^*}$
- So, search direction in $\mathbb{R}^n$:

$$\mathbf{p} = \begin{pmatrix} \mathbf{p_B} \\ \mathbf{p_N} \end{pmatrix} = \begin{pmatrix} -\mathbf{B}^{-1}\mathbf{N}_{\mathrm{J}^*} \\ \mathbf{e}_{\mathrm{J}^*} \end{pmatrix}$$

- Descent? Yes, because $\mathbf{c}^{\mathrm{T}}\mathbf{p} = \tilde{c}_{\mathrm{J}^*} < 0$!

The basis change (pivot), II

- Feasible? Must check that $\mathbf{Ap} = \mathbf{0}^m$ and that $p_i \geq 0$ if $x_i = 0$, $i \in B$. The first true by construction:
- (a) $\mathbf{Ap} = \mathbf{B}\mathbf{p_B} + \mathbf{N}\mathbf{p_N} = -\mathbf{B}\mathbf{B}^{-1}\mathbf{N}_{\mathrm{j}^*} + \mathbf{N}\mathbf{e}_{\mathrm{j}^*} = \mathbf{0}^m$
- (b) Suppose that $\mathbf{x_B} > \mathbf{0}^m$. Then, at least a small step in $\mathbf{p}$ keeps $\mathbf{x_B} \geq \mathbf{0}^m$

  But if there is an $\imath^*$ with $(\mathbf{x_B})_{\imath^*} = 0$ *and* $(\mathbf{p_B})_{\imath^*} < 0$ then it is not a feasible direction. (This happens because the current extreme point corresponds to more than one BFS—often a case of having redundant constraints)
- Must then perform a basis change without moving! A *degenerate basis change*: swap x_{j^*} for $x_{\imath^*}$ in the basis
- Otherwise (and normally), we utilize the unit direction corresponding to the increase from zero of the incoming basic variable

The basis change (pivot), III

- Line search? Linear objective; move the maximum step!
- Maximum step: If $\mathbf{p_B} \geq \mathbf{0}^m$ there is no finite maximum step! We have found an extreme direction $\mathbf{p}$ along which $\mathbf{c}^{\mathrm{T}}\mathbf{x}$ tends to $-\infty$! *Unbounded solution*
- Otherwise: choose a basic variable that first reaches zero (the "outgoing" variable): choose a variable $i \in B$ with minimum in

$$x_{\jmath^*} := \underset{i \in B}{\text{minimum}} \left\{ \frac{(\mathbf{B}^{-1}\mathbf{b})_i}{(\mathbf{B}^{-1}\mathbf{N}_{\jmath^*})_i} \,\middle|\, (\mathbf{B}^{-1}\mathbf{N}_{\jmath^*})_i > 0 \right\}$$

- Done. In the basis, replace $\imath^*$ by $\jmath^*$; go to pricing step
- If $x_{\jmath^*} = 0$ then the above corresponds to a "degenerate basis change"

Computational notes—how do we do all of this? I

- Given basis matrix $\mathbf{B}$, solve

$$\mathbf{B}\mathbf{x}_\mathbf{B} = \mathbf{b}$$

- Gives us BFS: $\mathbf{x}_\mathbf{B} = \mathbf{B}^{-1}\mathbf{b}$
- Pricing step: (a) Solve

$$\mathbf{B}^\mathrm{T}\mathbf{y} = \mathbf{c}_\mathbf{B} \quad \Longrightarrow \quad \mathbf{y}^\mathrm{T} = \mathbf{c}_\mathbf{B}^\mathrm{T}\mathbf{B}^{-1}$$

- (b) Calculate $\tilde{\mathbf{c}}_\mathbf{N}^\mathrm{T} = \mathbf{c}_\mathbf{N}^\mathrm{T} - \mathbf{y}^\mathrm{T}\mathbf{N}$, the reduced cost vector
- Choose the incoming variable, x_{j^*}
- Outgoing variable: Solve

$$\mathbf{B}\mathbf{p}_\mathbf{B} = -\mathbf{N}_{\mathrm{j}^*}$$

- Quotient rule for $(\mathbf{B}^{-1}\mathbf{b})_i/(-\mathbf{p}_\mathbf{B})_i$ gives outgoing variable, x_{i^*}, *and* value of the new basic variable, x_{j^*}

Computational notes—how do we do all of this? II

- Three similar linear systems in $\mathbf{B}$! LU factorization of $\mathbf{B}$ + three triangular substitutions
- Factorizations can be updated after basis change rather than done from scratch
- LP solvers like Cplex and XPRESS-MP have excellent numerical solvers for linear systems
- Linear systems the bulk of the work in solving an LP

Convergence

- *If all of the basic feasible solutions are non-degenerate, then the Simplex algorithm terminates after a finite number of iterations*
- *Proof:* (Rough argument) Non-degeneracy implies that the step length is > 0; hence, we cannot return to an old BFS once we have left it. There are finitely many BFSs
- Degeneracy: Can actually lead to cycling—the same sequence of BFSs is returned to indefinitely!
- Remedy: Change the incoming/outgoing criteria! Bland's rule: Sort variables according to some index ordering. Take the first possible index in the list. Incoming variable first in the list with the right sign of the reduced cost; outgoing variable the first in the list among the minima in the quotient rule

Initial BFS: Phase I of the Simplex method, I

- If a starting BFS cannot be found, do the following:
- Suppose $\mathbf{b} \geq \mathbf{0}^m$. Introduce *artificial variables* a_i in every row (or rows without a unit column)
- Solve the following Phase-I problem:

$$\begin{aligned}
\text{minimize} \quad w = \quad & (\mathbf{1}^m)^{\mathrm{T}}\mathbf{a} \\
\text{subject to} \quad & \mathbf{A}\mathbf{x} + \mathbf{I}^m\mathbf{a} = \mathbf{b}, \\
& \mathbf{x} \geq \mathbf{0}^n, \\
& \mathbf{a} \geq \mathbf{0}^m
\end{aligned}$$

- Possible cases: (a) $w^* = 0$, meaning that $\mathbf{a}^* = \mathbf{0}^m$ must hold. There is then a BFS in the *original* problem

Initial BFS: Phase I of the Simplex method, II

- Start Phase-II (i.e., to solve the original problem) from this BFS
- (b) $w^* > 0$. The optimal basis then has some $a_i^* > 0$; due to the objective function construction, there exists no BFS in the original problem. The problem is infeasible!
- What to do then? Modelling errors? Can be detected from the optimal solution. In fact, some LP problems are pure feasibility problems