Lecture 10: LP duality/sensitivity

Michael Patriksson

2010-11-24

The canonical primal–dual pair

$\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{c} \in \mathbb{R}^n$

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^{\mathrm{T}} \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}^n \end{aligned} \tag{1}$$

and

$$\begin{aligned} \text{minimize} \quad & w = \mathbf{b}^{\mathrm{T}} \mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^{\mathrm{T}} \mathbf{y} \geq \mathbf{c}, \\ & \mathbf{y} \geq \mathbf{0}^m \end{aligned} \tag{2}$$

The dual of the LP in standard form

$$\begin{aligned} \text{minimize} \quad & z = \mathbf{c}^{\mathrm{T}} \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}^n \end{aligned} \tag{P}$$

and

$$\begin{aligned} \text{maximize} \quad & w = \mathbf{b}^{\mathrm{T}} \mathbf{y} \\ \text{subject to} \quad & \mathbf{A}^{\mathrm{T}} \mathbf{y} \leq \mathbf{c}, \\ & \mathbf{y} \quad \text{free} \end{aligned} \tag{D}$$

Rules for formulating dual LPs, I

- We say that an inequality is *canonical* if it is of $\leq$ [respectively, $\geq$] form in a maximization [respectively, minimization] problem
- We say that a variable is *canonical* if it is ≥ 0
- The rule is that the dual variable [constraint] for a primal constraint [variable] is canonical if the other one is canonical. If the direction of a primal constraint [sign of a primal variable] is the opposite from the canonical, then the dual variable [dual constraint] is also opposite from the canonical
- Further, the dual variable [constraint] for a primal equality constraint [free variable] is free [an equality constraint]

Rules for formulating dual LPs, II

- Summary:

primal/dual constraint		**dual/primal variable**
canonical inequality	$\iff$	≥ 0
non-canonical inequality	$\iff$	≤ 0
equality	$\iff$	unrestricted (free)

Weak Duality Theorem

- *If* $\mathbf{x}$ *is a feasible solution to* (P) *and* $\mathbf{y}$ *a feasible solution to* (D), *then* $\mathbf{c}^{\mathrm{T}}\mathbf{x} \geq \mathbf{b}^{\mathrm{T}}\mathbf{y}$
- Similar relation for the primal–dual pair (2)–(1): the max problem never has a higher objective value
- *Proof.*
- Corollary: If $\mathbf{c}^{\mathrm{T}}\mathbf{x} = \mathbf{b}^{\mathrm{T}}\mathbf{y}$ for a feasible primal–dual pair $(\mathbf{x}, \mathbf{y})$ then they must be optimal

Strong Duality Theorem

- Strong duality is here established for the pair (P), (D)

- *If one of the problems* (P) *and* (D) *has a finite optimal solution, then so does its dual, and their optimal objective values are equal*

- *Proof.*

Complementary Slackness Theorem

- *Let* $\mathbf{x}$ *be a feasible solution to* (1) *and* $\mathbf{y}$ *a feasible solution to* (2). *Then* $\mathbf{x}$ *is optimal to* (1) *and* $\mathbf{y}$ *optimal to* (2) *if and only if*

$$x_j(c_j - \mathbf{y}^{\mathrm{T}}\mathbf{A}_{\cdot j}) = 0, \qquad j = 1, \dots, n, \qquad (3a)$$

$$y_i(\mathbf{A}_{i\cdot}\mathbf{x} - b_i) = 0, \qquad i = 1 \dots, m, \qquad (3b)$$

where $\mathbf{A}_{\cdot j}$ *is the* j^{th} *column of* $\mathbf{A}$, $\mathbf{A}_{i\cdot}$ *the* i^{th} *row of* $\mathbf{A}$

- *Proof.*

Necessary and sufficient optimality conditions, I

- We have seen above that the following statement characterizes the optimality of a primal–dual pair $(\mathbf{x}, \mathbf{y})$:

- $\mathbf{x}$ is feasible in (1), $\mathbf{y}$ is feasible in (2), and complementarity holds for the pair $(\mathbf{x}, \mathbf{y})$

Necessary and sufficient optimality conditions, II

- In other words, we have the following result (think of the KKT conditions):

- *Take a vector* $\mathbf{x} \in \mathbb{R}^n$*. For* $\mathbf{x}$ *to be an optimal solution to the linear program* (1)*, it is both necessary and sufficient that*
  (a) $\mathbf{x}$ *is a feasible solution to* (1)*;*
  (b) *corresponding to* $\mathbf{x}$ *there is a dual vector* $\mathbf{y} \in \mathbb{R}^m$ *such that* $\mathbf{x}$ *and* $\mathbf{y}$ *together satisfy the complementarity conditions* (3)*; and*
  (c) *the dual vector* $\mathbf{y}$ *is feasible in the dual problem* (2)

Necessary and sufficient optimality conditions, III

- This is precisely the same as the KKT conditions!
- Those who wish to establish this: note that there are no multipliers for the "$\mathbf{x} \geq \mathbf{0}^n$" constraints, and in the KKT conditions there are. Introduce such a multiplier vector, write down the KKT conditions, and see that the multiplier vector can later be eliminated
- Further: suppose that **x** and **y** are feasible respectively in (1) and (2). Then, the following are equivalent:
  (a) **x** and **y** have the same objective value;
  (b) **x** and **y** solve (1) and (2); and
  (c) **x** and **y** together satisfy complementarity

The Simplex method: global optimality conditions, I

- The Simplex method is remarkable in that it satisfies two of the three conditions at every BFS, and the remaining one is satisfied at optimality:

  (a) $\mathbf{x}$ is feasible after Phase-I has been completed

  (b) $\mathbf{x}$ and $\mathbf{y}$ always satisfy complementarity. Why? If x_j is in the basis, then it has a zero reduced cost, implying that the dual constraint j has no slack. If the reduced cost of x_j is non-zero (slack in dual constraint j), then its value is zero

The Simplex method: global optimality conditions, II

(c) The feasibility of $\mathbf{y}^{\mathrm{T}} = \mathbf{c}_{\mathbf{B}}^{\mathrm{T}}\mathbf{B}^{-1}$ is not fulfilled until we reach an optimal BFS. How is the incoming criterion related to this? We introduce as an incoming variable a variable which has the best reduced cost. Since the reduced cost measures the dual feasibility of $\mathbf{y}$, this means that we select the most violated dual constraint; at the new BFS, that constraint is then satisfied (since the reduced cost then is zero). The Simplex method hence works to try to satisfy dual feasibility by forcing a move such that the most violated dual constraint becomes satisfied!

Farkas' Lemma revisited

- *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$ *and* $\mathbf{b} \in \mathbb{R}^m$. *Then, exactly one of the systems*

$$\begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{b}, \\ \mathbf{x} &\geq \mathbf{0}^n, \end{aligned} \tag{I}$$

and

$$\begin{aligned} \mathbf{A}^{\mathrm{T}}\mathbf{y} &\leq \mathbf{0}^n, \\ \mathbf{b}^{\mathrm{T}}\mathbf{y} &> 0, \end{aligned} \tag{II}$$

has a feasible solution, and the other system is inconsistent

- *Proof.*

*Sensitivity analysis, I: Shadow prices are derivatives of a convex function!

- Suppose an optimal BFS is non-degenerate. Then, $\mathbf{c}^{\mathrm{T}}\mathbf{x}^* = \mathbf{b}^{\mathrm{T}}\mathbf{y}^* = \mathbf{c}_B^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{b}$ varies linearly as a function of $\mathbf{b}$ around its given value
- Non-degeneracy also implies that $\mathbf{y}^*$ is unique. Why?
- *Perturbation function* $\mathbf{b} \mapsto v(\mathbf{b})$ given by

$$v(\mathbf{b}) = \min \mathbf{c}^{\mathrm{T}}\mathbf{x} \quad = \max \mathbf{b}^{\mathrm{T}}\mathbf{y} \quad = \max_{k \in K} \mathbf{b}^{\mathrm{T}}\mathbf{y}_k$$
$$\text{s.t. } \mathbf{A}\mathbf{x} = \mathbf{b}, \qquad \text{s.t. } \mathbf{A}^{\mathrm{T}}\mathbf{y} \leq \mathbf{c}$$
$$\mathbf{x} \geq \mathbf{0}^n$$

  K: set of DFS, v is a piece-wise linear, convex function
- v convex (and finite in a neighbourhood of $\mathbf{b}$) implies that "v is differentiable at $\mathbf{b}$" $\iff$ "v has a unique subgradient there"
- Here: the derivative w.r.t. $\mathbf{b}$ is $\mathbf{y}^*$, that is, the change in the optimal value from a change in the right-hand side $\mathbf{b}$ equals the dual optimal solution

*Sensitivity analysis, II: Perturbations in data

- How to find a new optimum through re-optimization when data has changed:
- If an element of $\mathbf{c}$ changes, then the old BFS is feasible but may not be optimal. Check the new value of the reduced cost vector $\tilde{\mathbf{c}}$ and change the basis if some sign has changed
- If an element of $\mathbf{b}$ changes, then the old BFS is "optimal" but may not be feasible. Check the new value of the vector $\mathbf{B}^{-1}\mathbf{b}$ and change the basis if some sign has changed. Since the BFS is infeasible but "optimal", we use a dual version of the Simplex method: the *Dual Simplex method*:
- Find a negative basic variable $x_j \rightarrow$ outgoing basic variable x_s
- Choose among the non-basic variables for which the element $\mathbf{B}^{-1}\mathbf{N}_{sj} < 0$; means that the new basic variable will become positive
- Choose the incoming variable so that $\tilde{\mathbf{c}}$ keeps its sign