Lecture on Integer Programming

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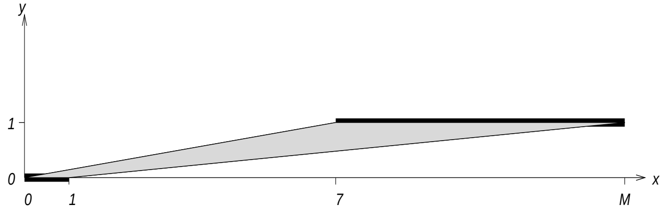
2010-12-01

When are integer models needed?

- Products or raw materials are indivisible
- Logical constraints: "if A then B"; "A or B"
- Fixed costs
- Combinatorics (sequencing, allocation)
- On/off-decision to buy, invest, hire, generate electricity, ...

Non polyhedral feasible sets

Either $0 \leq x \leq 1$ or $x \geq 7$

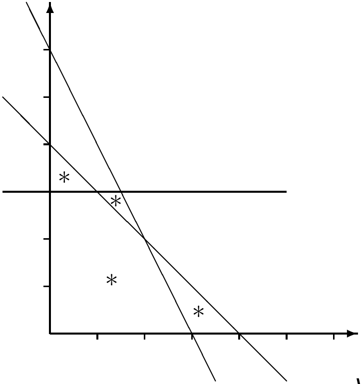


$$\text{Let } M \gg 1 : \quad x \leq 1 + My,\; x \geq 7y,\; y \in \{0, 1\}$$

Variable x may only take the values 2, 45, 78 & 107

$$x = 2y_1 + 45y_2 + 78y_3 + 107y_4$$
$$y_1 + y_2 + y_3 + y_4 = 1$$
$$y_1, y_2, y_3, y_4 \in \{0, 1\}$$

At least 2 of 3 constraints must be fulfilled



$*$ = feasible regions
$M \geq 2$

$$
\begin{aligned}
x_1 + x_2 &\leq 4 && (1)\\
2x_1 + x_2 &\leq 6 && (2)\\
x_2 &\leq 3 && (3)\\
x_1, x_2 &\geq 0
\end{aligned}
$$

$$
\begin{aligned}
x_1 + x_2 &\leq 4 + M(1 - y_1) && (1)\\
2x_1 + x_2 &\leq 6 + M(1 - y_2) && (2)\\
x_2 &\leq 3 + M(1 - y_3) && (3)\\
y_1 + y_2 + y_3 &\geq 2\\
y_1, y_2, y_3 &\in \{0, 1\}\\
x_1, x_2 &\geq 0
\end{aligned}
$$

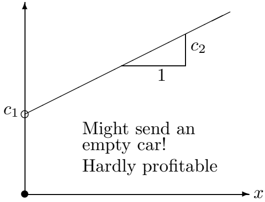
Fixed costs

$x =$ the amount of a certain product to be sent

If $x > 0$ then the initial cost c_1 (e.g. car hire) is generated

Variable cost c_2 per unit sent

Total cost: $f(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ c_1 + c_2 \cdot x & \text{if} \quad x > 0 \end{cases}$ effect wanted!



Let $M =$ car capacity

$$y = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$ effect wanted!

$$f(x, y) = c_1 \cdot y + c_2 \cdot x$$

$$x \leq M \cdot y$$ linear 0/1 model!

$$x \geq 0, \quad y \in \{0, 1\}$$

Other applications of integer optimization

- Facility location (new hospitals, shopping centers, etc.)
- Scheduling (on machines, personnel, projects, schools)
- Logistics (material- and warehouse control)
- Distribution (transportation of goods, buses for disabled persons)
- Production planning
- Telecommunication (network design, frequency allocation)

The combinatorial explosion

Assign n persons to carry out n jobs # feasible solutions: $n!$
Assume that a feasible solution is evaluated in 10^{-9} seconds

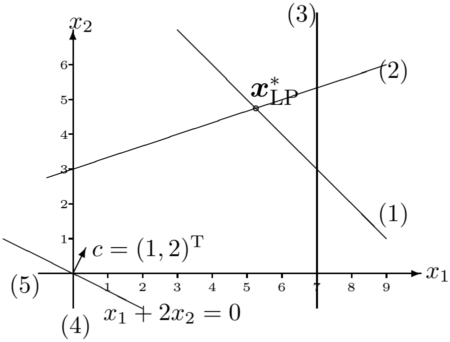
n	2	5	8	10	100
$n!$	2	120	$4.0 \cdot 10^4$	$3.6 \cdot 10^6$	$9.3 \cdot 10^{157}$
$\lceil \text{time} \rceil$	10^{-8} s	10^{-6} s	10^{-4} s	10^{-2} s	10^{142} yrs

Complete enumeration of all solutions is **not** an efficient algorithm!
An algorithm exists that solves this problem in time $\mathcal{O}(n^4) \propto n^4$

n	2	5	8	10	100	1000
n^4	16	625	$4.1 \cdot 10^3$	10^4	10^8	10^{12}
$\lceil \text{time} \rceil$	10^{-7} s	10^{-6} s	10^{-5} s	$10^{-5} s$	10^{-1} s	17 min

Linear continuous optimization model

$$
\begin{array}{lllllll}
\max & z_{\text{LP}} = & x_1 + 2x_2 & & & \\
\text{s.t.} & & x_1 + x_2 & \leq & 10 & (1) \\
& & -x_1 + 3x_2 & \leq & 9 & (2) \\
& & x_1 & \leq & 7 & (3) \\
& & x_1, x_2 & \geq & 0 & (4,5)
\end{array}
$$

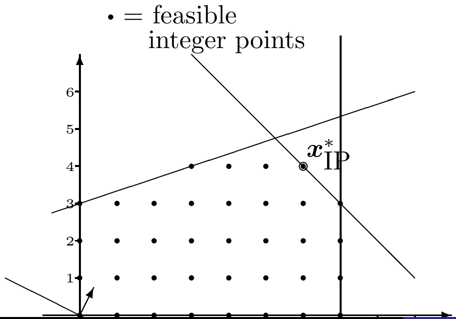


$$\mathbf{x}^*_{\text{LP}} = \begin{pmatrix} 21/4 \\ 19/4 \end{pmatrix}$$

$$z^*_{\text{LP}} = 14\tfrac{3}{4}$$

Linear integer optimization model

$$\begin{array}{llrcrcll}
\max & z_{IP} = & x_1 & + & 2x_2 & & & \\
\text{s.t.} & & x_1 & + & x_2 & \leq & 10 & (1) \\
& & -x_1 & + & 3x_2 & \leq & 9 & (2) \\
& & x_1 & & & \leq & 7 & (3) \\
& & & & x_1, x_2 & \geq & 0 & (4,5) \\
& & & & x_1, x_2 & & \text{integer} &
\end{array}$$



$$\mathbf{x}^*_{IP} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

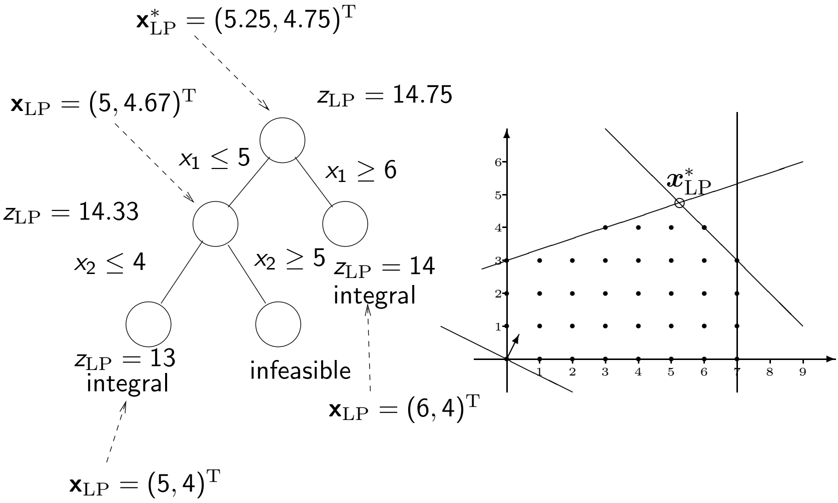
$$z^*_{IP} = 14 < z^*_{LP}$$

Classic methods

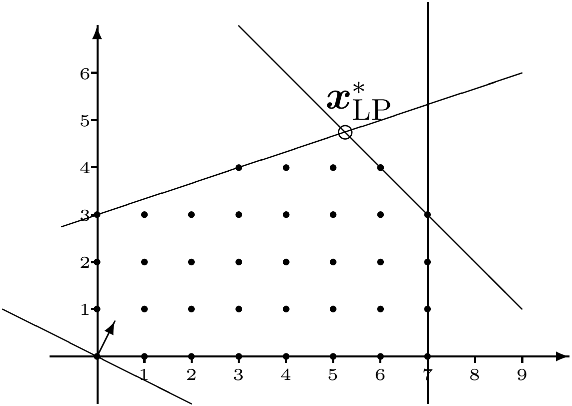
- *Branch–and–Bound*: relaxation plus divide–and–conquer
- *Cutting plane method*: relaxation plus generations of constraint that cut off infeasible (e.g., non-integer) points generated
- "Relaxation" can be the continuous or Lagrangian one
- *Lagrangian optimization*: Lagrangian relaxation plus multiplier optimization
- These methods are often combined (e.g., cutting planes added at nodes in B&B tree: Branch & Cut)

The branch–and–bound-algorithm

Relax integrality constraints ⇒ linear program ⇒



Cutting plane



Add a constraint such that no feasible points are cut off and the LP relaxation solves the IP. 2 min!

Cutting plane methods

- Goal: generate the convex hull of the feasible integer vectors
- Result: Can solve the IP by solving the LP relaxation over this convex hull
- Compare IP example: one extra linear constraint defines the entire convex hull! ($x_2 \leq 4$)
- Means: Relax problem (e.g., continuous relaxation); Solve. If infeasible solution, generate constraint to the relaxation that cuts off that vector but no feasible vectors. Repeat
- Constraint generation called a *separation oracle*

The complexity of integer optimization, I

The Mexico LP has (in the version which is handed out) 113 variables and 84 linear constraints. Solution by a slow (333 MHz Unix) computer: 0.01 s

IP variant: add a fixed cost for using a railway link for the raw material transport $\Longrightarrow$ 78 0/1 variables

The complexity of integer optimization, I

Fixed cost	100 $\Rightarrow$	20 s.	
		18,000 B & B nodes	
		60,000 simplex iterations	
	300 $\Rightarrow$	3 min.	
		208,000 B & B nodes	
		650,000 simplex iterations	

The complexity of integer optimization, I

Note:

- There are $2^{78} \approx 0.3 \cdot 10^{24}$ possible combinations. B & B is good at implicitly enumerating them all ...
- The higher the fixed cost, the more difficult the problem. Why?
  Continuous relaxation worse approximation

The complexity of integer optimization, II

- Knapsack problem:

$$\text{minimize} \sum_{i=1}^{n} c_i x_i$$
$$\text{subject to} \sum_{i=1}^{n} a_i x_i \leq b$$
$$x_i \in \{0, 1\} \qquad i = 1, \ldots, n.$$

- AMPL model: `var x{1..5} integer, >=0;`

```
maximize
ka:213*x[1]-1928*x[2]-11111*x[3]-2345*x[4]+9123*x[5];
subject to c1:
12223*x[1]+12224*x[2]+36674*x[3]+61119*x[4]+85569*x[5]
= 89643482;
```

The complexity of integer optimization, II

- The problem has an optimal solution
- LP relaxation trivial: sort variables in descending order of c_j/a_j; take the best one; fill the knapsack exactly
- LP solution? $\mathbf{x}^*_{\mathrm{LP}} \approx (0, 0, 0, 0, 1047.62)^{\mathrm{T}}$
- Optimal solution: $\mathbf{x}^* = (7334, 0, 0, 0, 0)^{\mathrm{T}}$
- IP solution? Report after roughly 3 hours of CPU time:

```
Mixed integer rounding cuts applied:  2
Gomory fractional cuts applied:  1
CPLEX 10.1.0:  unrecoverable failure with no
integer solution.
20298576 MIP simplex iterations
384198302 branch-and-bound nodes; no basis.
x [*] := 0 0 0 0 0 ;
```

The Philips example—TSP solved heuristically

- Let c_{ij} denote the distance between cities i and j, with

$$\begin{aligned} \{i,j\} \subset \mathcal{N} - & \text{ set of nodes} \\ (i,j) \in \mathcal{L} - & \text{ set of links} \end{aligned}$$

- Links (i,j) and (j,i) the same; direction plays no role
- $x_{ij} = \begin{cases} 1, & \text{if link } (i,j) \text{ is part of the TSP tour,} \\ 0, & \text{otherwise} \end{cases}$

The Philips example—TSP solved heuristically

- The Traveling Salesman Problem (TSP):

$$\begin{array}{llll}
\text{minimize} & \displaystyle\sum_{(i,j)\in\mathcal{L}} c_{ij}x_{ij} & & \\
\text{subject to} & \displaystyle\sum_{(i,j)\in\mathcal{L}:\{i,j\}\subset S} x_{ij} \leq |\mathcal{S}|-1, & \mathcal{S}\subset\mathcal{N}, & (1)\\
& \displaystyle\sum_{(i,j)\in\mathcal{L}} x_{ij} = n, & & (2)\\
& \displaystyle\sum_{i\in\mathcal{N}:(i,j)\in\mathcal{L}} x_{ij} = 2, & j\in\mathcal{N}, & (3)\\
& x_{ij}\in\{0,1\}, & (i,j)\in\mathcal{L} &
\end{array}$$

Interpretations

- Constraint (2) implies that in total n cities must be visited;
- Constraint (3) implies that each city is connected to two others, such that we make sure to arrive from one city and leave for the next
- Construct a solution which satisfies (2), (3) but not (1). 2 min !
- Constraint (1) implies that there can be no *sub-tours*, that is, a tour where fewer than n cities are visited (that is, if $\mathcal{S} \subset \mathcal{N}$ then there can be at most $|\mathcal{S}| - 1$ links between nodes in the set $\mathcal{S}$, where $|\mathcal{S}|$ is the cardinality–number of members of–the set $\mathcal{S}$);

Lagrangian relaxation

- TSP is NP-hard—no known polynomial algorithms exist
- Lagrangian relax (3) for all nodes except starting node
- Find a solution that satisfies (1), (2) but not (3), 2 min !
- Remaining problem: 1-MST—find the minimum spanning tree in the graph without the starting node and its connecting links; then, add the two cheapest links to connect the starting node
- Starting node $s \in \mathcal{N}$ and connected links assumed removed from the graph

Lagrangian relaxation

- Objective function of the Lagrangian problem:

$$q(\boldsymbol{\lambda}) = \underset{\mathbf{x}}{\text{minimum}} \sum_{(i,j)\in\mathcal{L}} c_{ij}x_{ij} + \sum_{j\in\mathcal{N}} \lambda_j \Big(2 - \sum_{i\in\mathcal{N}:(i,j)\in\mathcal{L}} x_{ij}\Big)$$
$$= 2\sum_{j\in\mathcal{N}} \lambda_j + \underset{\mathbf{x}}{\text{minimum}} \sum_{(i,j)\in\mathcal{L}} (c_{ij} - \lambda_i - \lambda_j)x_{ij}$$

- A high (low) value of the multiplier λ_j makes node j attractive (unattractive) in the 1-MST problem, and will therefore lead to more (less) links being attached to it
- Subgradient method for updating the multipliers

Lagrangian relaxation

- Updating step:

$$\lambda_j := \lambda_j + \alpha \left(2 - \sum_{i \in \mathcal{N} : (i,j) \in \mathcal{L}} x_{ij} \right), \qquad j \in \mathcal{N},$$

  where $\alpha > 0$ is a step length

- Update means:

  Current degree at node j :

$$\begin{cases} > 2 \implies \lambda_j \downarrow \text{(link cost } \uparrow) \\ = 2 \implies \lambda_j \leftrightarrow \text{(link cost constant)} \\ < 2 \implies \lambda_j \uparrow \text{(link cost } \downarrow) \end{cases}$$

- Link cost shifted upwards (downwards) if too many (too few) links connected to node j in the 1-MST

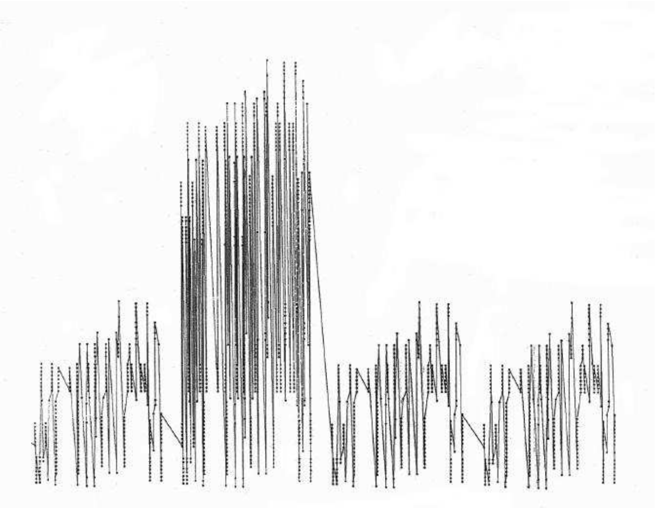
Feasibility heuristic

- Adjusts Lagrangian solution $\mathbf{x}$ such that the resulting vector is feasible
- Often a good thing to do when approaching the dual optimal solution—$\mathbf{x}$ often then only mildly infeasible
- Identify path in 1-MST with many links; form a subgraph with the remaining nodes which is a path; connect the two
- Result: A Hamiltonian cycle (TSP tour)
- We then have both an upper bound (feasible point) and a lower bound (q) on the optimal value—a quality measure: $[f(\mathbf{x}) - q(\boldsymbol{\mu})]/q(\boldsymbol{\mu})$

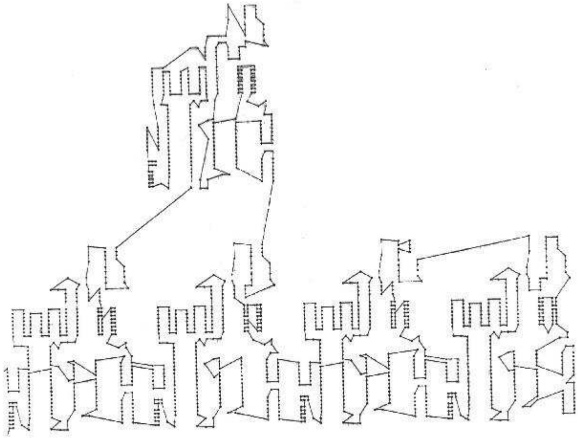
The Philips example

- Fixed number of subgradient iterations
- Feasibility heuristic used every K iterations ($K > 1$), starting at a late subgradient iteration
- Typical example: Optimal path length in the order of 2 meters; upper and lower bounds produced concluded that the relative error in the production plan is *less than 7 %*
- Also: increase in production by some 70 %

Initial drill pattern



Near-optimal drill pattern

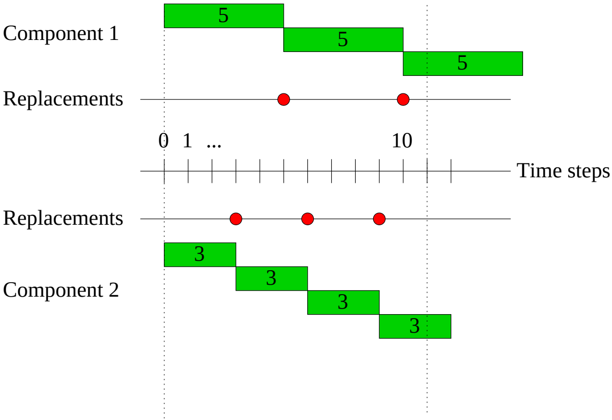


A motivating example

- Wind power turbine with 14 major components
- Crane is necessary for replacement of failed components
- Given failure of one component (opportunity), decide if other components should be replaced
- The decision is based on:
  - Components' life distributions (data)
  - Price of new component and maintenance occasion cost
  - Remaining life of the turbine

Deterministic component lives

cost: $2c_1 + 3c_2 + 5d$



Definition

Given lives T_i for every component i, costs c_{it}, d and timehorizon T, minimize the

The variables

$$x_{it} = \begin{cases} 1 & \text{component } i \text{ is replaced at time } t, \\ 0 & \text{otherwise.} \end{cases}$$

$$z_t = \begin{cases} 1 & \text{maintenance performed at time } t, \\ 0 & \text{otherwise.} \end{cases}$$

The model

$$\begin{aligned} \text{minimize} \quad & \sum_{t=1}^{T} \left(\sum_{i \in \mathcal{N}} c_i x_{it} + d z_t \right) \\ \text{subject to} \quad & \sum_{t=\ell}^{T_i+\ell-1} x_{it} \geq 1, \quad \ell = 1, \ldots, T - T_i, \quad i \in \mathcal{N} \\ & x_{it} \leq z_t, \quad t = 1, \ldots, T, \quad i \in \mathcal{N} \\ & x_{it}, z_t \in \{0, 1\}, \quad t = 1, \ldots, T, \quad i \in \mathcal{N} \end{aligned}$$

- In fact the requirement $x_{it} \in \{0, 1\}$ can be replaced by $x_{it} \in [0, 1]$—it will be integral at an optimal solution!

Numerical example

Let

$$T = 60, \quad N = 4,$$
$$T_1 = 13, \quad T_2 = 19, \quad T_3 = 34, \quad T_4 = 18,$$
$$c_1 = 80, \quad c_2 = 185, \quad c_3 = 160, \quad c_4 = 125$$

Numerical Example

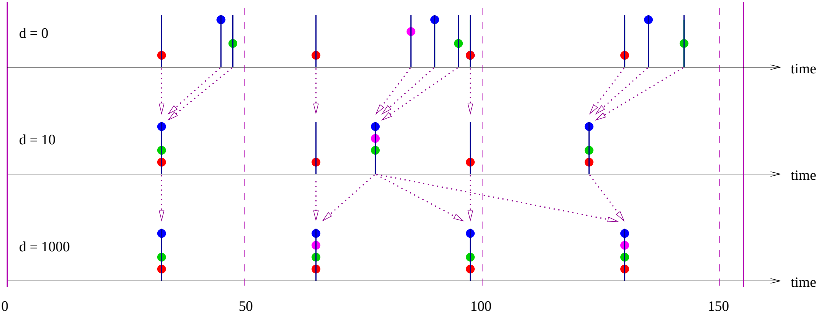


Figure: Optimal solutions for three cases of fixed costs

Case study: Volvo Aero, Trollhättan

- Maintenance of the RM12 engine (JAS and civil aircraft)
- The engine consists of several modules; parts in modules are either safety-critical (typically rotating ones) or on-condition
- Safety-critical parts have fixed lives (deterministic); also others are monitored and are considered stochastic—conditional lives
- Goal: Maintain the whole fleet such that the total maintenance cost is minimized (or the total "time on wing" is maximized)

Status and goals for the future

- Current status:
  - System may be implemented at Volvo Aero
  - Collaborations: Energy and Environment, Chalmers (nuclear/wind power production) and CHARMEC (railway)
- Create optimization model and solution methodology system for general cases and systems
- New PhD student (Emil)
- Research rewarded "stora produktivitetspriset"
- Searching for masters students!