Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MAN280
OPTIMIZATION, BASIC COURSE

Date:	08–12–15
Time:	House V, morning
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Peter Lindroth (0762-721860)
Result announced:	09–01–16 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{aligned}
\text{minimize} \quad & z = -x_1 + x_2, \\
\text{subject to} \quad & -x_1 + 2x_2 \geq 1/2, \\
& -2x_1 - 2x_2 \geq 1, \\
& x_1 \in \mathbb{R} \text{ (free)}, \\
& x_2 \geq 0.
\end{aligned}$$

(2p) a) Solve this problem by using phase I and phase II of the simplex method.

[*Aid:* Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

for producing basis inverses.]

(1p) b) Without solving the dual to the problem above, motivate clearly whether there are no optimal dual solutions, a unique optimal dual solution (if so, present it) or multiple optimal dual solutions (if so, present at least two of them).

(3p) Question 2

(convergence of an exterior penalty method)

Let us consider a general optimization problem:

$$\begin{aligned} &\text{minimize } f(\boldsymbol{x}), \\ &\text{subject to } \boldsymbol{x} \in S, \end{aligned} \tag{1}$$

where $S \subset \mathbb{R}^n$ is a non-empty, closed set and $f : \mathbb{R}^n \to \mathbb{R}$ is a given differentiable function. We assume that the feasible set S of the optimization problem (1) is given by the system of inequality and equality constraints:

$$\begin{aligned} S = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid g_i(\boldsymbol{x}) &\leq 0, \quad i = 1, \ldots, m, \\ h_j(\boldsymbol{x}) &= 0, \quad j = 1, \ldots, \ell \,\}, \end{aligned} \tag{2}$$

where $g_i \in C(\mathbb{R}^n)$, $i = 1, \ldots, m$, $h_j \in C(\mathbb{R}^n)$, $j = 1, \ldots, \ell$.

We choose a function $\psi : \mathbb{R} \to \mathbb{R}_+$ such that $\psi(s) = 0$ if and only if $s = 0$ (typical examples of $\psi(\cdot)$ are $\psi_1(s) = |s|$, or $\psi_2(s) = s^2$), and introduce the function

$$\nu \check{\chi}_S(\boldsymbol{x}) := \nu \Bigg(\sum_{i=1}^{m} \psi\big(\max\{0, g_i(\boldsymbol{x})\}\big) + \sum_{j=1}^{\ell} \psi\big(h_j(\boldsymbol{x})\big) \Bigg), \tag{3}$$

where the real number $\nu > 0$ is called a *penalty parameter*.

We assume that for every $\nu > 0$ the approximating optimization problem to

$$\text{minimize } f(\boldsymbol{x}) + \nu \check{\chi}_S(\boldsymbol{x}) \tag{4}$$

has at least one optimal solution $\boldsymbol{x}_\nu^*$.

We then have the following result.

THEOREM 1 *Assume that the original constrained problem* (1) *possesses optimal solutions. Then, every limit point of the sequence* $\{\boldsymbol{x}_\nu^*\}$, $\nu \to +\infty$, *of globally optimal solutions to* (4) *is globally optimal in the problem* (1).

Prove this theorem.

(3p) Question 3

(applications of Weierstrass' Theorem)

For each of the following functions f_i, $i = 1, 2, 3$, motivate carefully if a global minimum is attained on the corresponding set S_i, $i = 1, 2, 3$.

(1) $f_1(\boldsymbol{x}) = -e^{-\frac{(x_1+2)^2+(x_2+1)^2}{10}} + 10e^{-\frac{(x_1+2)^2+(x_2+1)^2}{100}} + \frac{1}{50}\left((x_1+2)^2 + (x_2+1)^2\right) + \frac{1}{10}x_1,$
$S_1 = \mathbb{R}^2$.

(2) $f_2(\boldsymbol{x}) = \begin{cases} -\frac{1}{x_1^2+(x_2-1)^2+2x_3^2} + x_3^2, & \text{if } x_1 > 0, \\ 0, & \text{if } x_1 \leq 0, \end{cases}$
$S_2 = \left\{\boldsymbol{x} \in \mathbb{R}^3 \mid -5 \leq x_i \leq 5, \ \forall i\right\}.$

(3) $f_3(\boldsymbol{x}) = \left(x_1 + x_2^2\right)^2 + x_1 + 3x_2 + 200,$
$S_3 = \mathbb{R}^2$.

Question 4

(modeling)

The government has assigned you to lead their aid program. They are willing to spend 1 % of the national gross product of b SEK. They are considering to give aid to a set of countries $\mathcal{N} = \{1, \dots, N\}$. The aim of the aid program is to increase the Human Development Index (HDI) in the countries, which is calculated by measuring the three factors education per capita, life expectancy and gross national product per capita (GDP). We may therefore consider the HDI index as a measure of *development per capita* in a country.

For all countries $j \in \mathcal{N}$ let a_j denote the current value of the HDI index, c_j the increase of HDI per SEK given as aid to the country and p_j the population size. There are only a limited number of aid programs in each of the countries. This puts a limit on the maximal aid that a country can receive, let d_j SEK denote the maximal aid country j can receive.

(1p) a) Write a linear program for distributing aid that maximizes the total HDI in the region formed by all the countries considered.

(1p) b) You are given new directives from the government: they think that the aid should be focused on a maximal number of M countries. Introduce integer variables in your model (i.e., write an integer linear programming model) in order to accommodate this demand.

(1p) c) There has recently been some discussions concerning the aid to countries with a high HDI. The government wants you to write a new model that maximizes the minimal HDI among the countries. Extend your model in a) to an LP model that accommodates this demand.

(3p) Question 5

(the Frank–Wolfe method)

Consider the optimization problem to

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := x_1^2 + x_1x_2 + 2x_2^2 - 10x_1 - 4x_2, \\ \text{subject to} \quad & x_1 + x_2 \leq 3, \\ & 0 \leq x_1 \leq 2, \quad 0 \leq x_2 \leq 2. \end{aligned}$$

Start at the point $\boldsymbol{x}^0 = (0,0)^{\mathrm{T}}$ and perform two iterations of the Frank–Wolfe method. (Recall that the Frank–Wolfe method starts at some feasible point. Given an iteration k and feasible iterate $\boldsymbol{x}_k$ it produces a feasible search direction $\boldsymbol{p}_k$ through the minimization of the first-order Taylor expansson of f at $\boldsymbol{x}_k$. The next iterate is found through an exact line search in f along the search direction, such that the resulting vector is also feasible.) Write out the upper and lower bounds for the optimal objective function value that the algorithm generates in each iteration, and give a theoretical motivation to them. If an optimum is found, state so, and motivate why it is an optimum.

(3p) Question 6

(convex problem)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := -\ln(x_1 + x_2) + x_3 \ln x_3, \\ \text{subject to} \quad & g_i(\boldsymbol{x}) := -x_i + 1 \leq 0, \qquad i = 1,2,3, \\ & g_4(\boldsymbol{x}) := -x_1 + 2x_2^2 + 4x_3^2 - 10 \leq 0. \end{aligned}$$

Establish whether this is a convex problem or not.

[*Note:* By "convex problem" we refer to the property that the objective function is convex in a minimization problem, and that the feasible set is a convex set.]

(3p) Question 7

(linear programming duality)

Consider the following two polyhedral sets corresponding to the feasible sets of the standard pair of primal–dual linear programs:

$$
\begin{aligned}
X &= \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}^n \,\}, \\
Y &= \{\, \boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{c}, \quad \boldsymbol{y} \geq \boldsymbol{0}^m \,\}.
\end{aligned}
$$

Prove that if both X and Y are non-empty, then at least one of them must be unbounded.

[*Remark:* This result can in fact be strengthened to the following: If at least one of the sets X and Y is non-empty, then at least one of them is non-empty and unbounded; this result is due to Clark (1961).]

Good luck!