Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MAN280
OPTIMIZATION, BASIC COURSE

Date:	11–04–26
Time:	House V, 8^{30}–12^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Adam Wojciechowski (0703-088304)
Result announced:	11–05–10 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{aligned}
\text{maximize} \quad & c_1x_1 + c_2x_2 \\
\text{subject to} \quad & x_1 + 2x_2 \leq 4, \\
& x_1 - 2x_2 \geq -2, \\
& x_1 \geq 0, \\
& x_2 \geq 0.
\end{aligned}$$

(2p) a) Solve this problem for $c_1 = 1$ and $c_2 = 4$ using phase I (if necessary) and phase II of the simplex method.

Aid: Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(1p) b) For which costs $\boldsymbol{c} = (c_1, c_2)^{\mathrm{T}}$ is the solution optimal?

[*Hint:* Use a graphical representation of the problem and illustrate the answer to the question therein.]

Motivate your answer!

(3p)

Question 2

(the Separation Theorem)

The Separation Theorem can be stated as follows.

Suppose that the set $S \subseteq \mathbb{R}^n$ is closed and convex, and that the point $\boldsymbol{y}$ does not lie in S. Then, there exist a vector $\boldsymbol{\pi} \neq \mathbf{0}^n$ and $\alpha \in \mathbb{R}$ such that $\boldsymbol{\pi}^{\mathrm{T}}\boldsymbol{y} > \alpha$ and $\boldsymbol{\pi}^{\mathrm{T}}\boldsymbol{x} \leq \alpha$ for all $\boldsymbol{x} \in S$.

Establish the theorem using basic results from the course. If you rely on other results when performing your proof of the above theorem, then those results must be stated; they may however be utilized without proof.

Question 3

(descent and optimality in optimization)

(1p) a) Consider the function

$$f(\boldsymbol{x}) := x_1^2 - x_1 x_2 + 5x_2^3 - 12.$$

At $\boldsymbol{x} = (1, 1)^{\mathrm{T}}$, is the vector $\boldsymbol{p} = (1, -2)^{\mathrm{T}}$ a direction is descent?

(2p) b) A continuously differentiable function f has a local minimum at the origin, that is, at $\boldsymbol{x}^* = \mathbf{0}^2$, subject to the constraints that $x_1 - x_2 \leq 0$ and $2x_1 + x_2 \leq 0$. Determine the possible values of $\nabla f(\boldsymbol{x})$ at $\boldsymbol{x}^* = \mathbf{0}^2$.

(3p) Question 4

(optimality conditions)

Consider the problem to

$$\begin{aligned} \text{maximize} \quad & f(\boldsymbol{x}) := \boldsymbol{b}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \sum_{j=1}^{n} x_j^2 \leq 1, \end{aligned}$$

where $\boldsymbol{b} \neq \mathbf{0}^n$ is a given vector.

Show that the unique globally optimal solution is $\boldsymbol{x}^* = \boldsymbol{b}/\|\boldsymbol{b}\|$.

Question 5

(modeling)

Consider a company that sell a product which is imported from a set of producers, $\{1, \ldots, n\}$. Your assignment is to plan the import of the company over a planning period from time 0 to time T. Let c_{jt} be the price per unit product from producer j at time t and let k_{jt} be the maximum amount that can be imported from producer j at time t. Note that if products are imported from a producer at time

t, they will arrive at the company at time $t+1$. The demand for the product at time t is d_t units ($d_0 = 0$). The company has a warehouse where it can store at most M units between each time step, at a cost of f per unit.

(2p) a) Formulate a linear optimization model for the minimization of the cost of importing the product, while fulfilling the demand at each time step.

(1p) b) You are told that there is a possibility of not fulfilling the demand at each time. If the company does not fulfill the demand at some time step, a cost γ per unit shortage has to be paid. Reformulate the model to include this additional fact.

(3p) **Question 6**

(the gradient projection algorithm)

The gradient projection algorithm is a generalization of the steepest descent method to problems over convex sets. Given a feasible point $\boldsymbol{x}^k$, the next point is obtained according to $\boldsymbol{x}^{k+1} = \operatorname{Proj}_X\left(\boldsymbol{x}^k - \alpha_k \nabla f(\boldsymbol{x}^k)\right)$, where X is the convex set over which we minimize, $\alpha_k > 0$ is the step length, and $\operatorname{Proj}_X(y) = \arg\min_{\boldsymbol{x}\in X} \|\boldsymbol{x} - \boldsymbol{y}\|$ denotes the closest point to y in X.

Consider the problem to

$$
\begin{aligned}
\underset{\boldsymbol{x}}{\text{minimize}} \quad & f(\boldsymbol{x}) := x_1^2 + 2x_2^2 - 2x_1x_2 - 2x_1 - 3x_2, \\
\text{subject to} \quad & 0 \leq x_1 \leq 3, \\
& 0 \leq x_2 \leq 2.
\end{aligned}
$$

Start at the point $\boldsymbol{x}^0 = (0,0)^{\mathrm{T}}$ and perform two iterations of the gradient projection algorithm using step length $\alpha_k = 1$ for all k. You may solve the projection problem in the algorithm graphically. Is the point obtained a global/local minimum? Motivate why/why not.

Question 7

(short questions)

Answer these short questions. You must motivate your answers; an answer without motivation results in zero points.

(1p) a) Consider the following problem:

$$\begin{aligned} \text{maximize} \quad & f(\boldsymbol{x}), \\ \text{subject to} \quad & g_i(\boldsymbol{x}) \leq 0, \quad i \in \{1, \dots, m\}, \\ & \boldsymbol{x} \in \mathbb{R}^n. \end{aligned}$$

Assume that f and g_i, $i \in \{1, \dots, m\}$, are convex, differentiable functions. If $\hat{\boldsymbol{x}}$ is a KKT point for the above problem, can we conclude that $\hat{\boldsymbol{x}}$ is optimal? If not, give a counter-example!

(2p) b) Consider the primal problem to

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}), && \text{(1a)} \\ \text{subject to} \quad & x_1^2 + x_2^2 \leq 4, && \text{(1b)} \\ & x_1 + x_2 \geq 1, && \text{(1c)} \\ & (x_1, x_2) \in X, && \text{(1d)} \end{aligned}$$

where $X \subset \mathbb{R}^2$ is a closed set.

We formulate the Lagrangian dual problem by performing a Lagrangian relaxation of constraints (1b) and (1c), and denote the dual objective function by q. We solve the Lagrngian dual problem with an algorithm that produces a sequence of points converging to a dual point $\hat{\boldsymbol{\mu}}$, which we assume is a KKT point in the Lagrangian dual problem. Can we conclude anything about the optimal objective value of the dual problem? Can we conclude anything about the optimal objective value of the primal problem? Can we conclude something more if X is a convex set and f is a convex function?