Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MAN280
OPTIMIZATION, BASIC COURSE

Date:	12–08–30
Time:	House V, morning, 8^{30}–13^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Hossein Raufi (0703-088304)
Result announced:	12–09–14 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program to

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 - 3x_2 \geq -6, \\ & x_1 - x_2 \leq 1, \\ & x_1 \geq 0, \\ & x_2 \geq 0. \end{aligned}$$

(2p) a) Solve this problem using phase I (so that you begin with a unit matrix as the first basis) and phase II of the simplex method.

Aid: Utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(1p) b) If an optimal solution exists, use your calculations to decide if it is unique. If the problem is unbounded, use your calculations to specify a direction of unboundedness.

(3p)

Question 2

(Newton's method)

Consider the unconstrained problem to

$$\begin{aligned} \text{minimize} \quad & f(x, y) := (x-1)^3 + y^2, \\ \text{subject to} \quad & (x, y) \in \mathbb{R}^2. \end{aligned}$$

Assume that we use Newton's method with unit step length on the above problem, and let (x_0, y_0) be the starting point, where $x_0 \neq 1$. Show that after k iterations we obtain the point

$$(x_k, y_k) = \left(1 + \frac{1}{2^k}(x_0 - 1), 0\right).$$

Does Newton's method converge to a local or global optimal solution? Motivate you answer!

Question 3

(sufficient global optimality conditions)

(1p) a) Consider an optimization problem of the following general form:

$$\begin{aligned} &\text{minimize } f(\boldsymbol{x}), && \text{(1a)}\\ &\text{subject to } \boldsymbol{x} \in S, && \text{(1b)} \end{aligned}$$

where $S \subseteq \mathbb{R}^n$ is nonempty, closed and convex, and $f : \mathbb{R}^n \to \mathbb{R}$ is convex on S.

Establish the fact that if a vector $\boldsymbol{x}^* \in S$ is a local minimum in this problem then it is also a global optimum.

(2p) b) Suppose further that S is described by $S := \{\, x \in \mathbb{R}^n \mid g_i(x) \leq 0,\ i = 1, \dots, m \,\}$, where $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$, are differentiable, convex functions. State the Karush–Kuhn–Tucker conditions for $\boldsymbol{x}^* \in S$ to be a local optimum in the problem (1) and establish that these conditions are in fact *sufficient* for $\boldsymbol{x}^*$ to be a *global* optimum in the problem (1).

Question 4

(convexity)

Prove or disprove the following three claims.

(1p) a) $f(x_1, x_2, x_3) = x_1^4 + x_2^2 + 4x_2x_3 + 5x_3^2$ is convex for all $\boldsymbol{x} \in \mathbb{R}^3$.

(1p) b) $f(x_1, x_2) = \max\{2x_1 - x_2, x_2^2\}$ is convex for all $\boldsymbol{x} \in \mathbb{R}^2$.

(1p) c) $f(x_1, x_2) = 2(x_1^3 + x_2^3) + x_1^2x_2^2 + 4x_2^2$ is convex near $\boldsymbol{x} = (0, 0)^{\mathrm{T}}$ (i.e., there is a small ball around the origin in which the function is convex).

(3p) Question 5

(linear programs)

With $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$, and $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, the primal LP problem, denoted (P), is the following:

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned}$$

State the LP dual of (P), and describe a polyhedron which equals the set of primal–dual optimal solutions.

[Note:] The polyhedron should be described in terms of the *original data* $\boldsymbol{A}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ only.

(3p) Question 6

(modelling)

Consider a graph $G = (V, E)$ with node set V and edge set E. You are required to send d units of flow from the source $s \in V$ to the sink $t \in V$. Each edge $(i, j) \in E$ has a capacity c_{ij}. The cost for sending x units of flow through edge $(i, j) \in E$ is $f_{ij}(x)$, where

$$f_{ij}(x) = \max\{m^1_{ij} + k^1_{ij}x, \dots, m^n_{ij} + k^n_{ij}x\},$$

i.e., the maximum of n affine functions. Formulate the described cost-minimizing problem as a linear program.

[Note:] Do *not* solve the problem.

(3p) Question 7

(Lagrangian duality)

Consider the following quadratic programming problem:

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := x_1^2 + 2x_2^2 - 4x_1, \\ \text{subject to} \quad & 2x_1 + x_2 \leq 2, \\ & x_j \geq 0, \quad j = 1, 2. \end{aligned} \tag{1}$$

We will attack this problem by using Lagrangian duality.

a) Consider Lagrangian relaxing the complicating constraint (1). Write down explicitly the resulting Lagrangian subproblem of minimizing the Lagrange function over the remaining constraints. Construct an explicit formula for the Lagrangian dual function. Establish that it is a concave function.

b) Solve the Lagrangian dual problem. State in particular the optimal solution, and the optimal value of the dual objective function.

c) Utilize the result in b) to generate an optimal solution to the original, primal, problem. Verify that strong duality holds. Is the primal optimal solution unique?