Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MMG621
OPTIMIZATION, BASIC COURSE

Date:	13–04–02
Time:	House V, morning, 8^{30}–13^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3.
	Questions are *not* numbered by difficulty.
	To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Jakob Hultgren (0703-088304)
Result announced:	13–04–18
	Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program to

$$
\begin{aligned}
\text{minimize} \quad & x_1 - 2x_2, && \text{(1a)}\\
\text{subject to} \quad & -x_1 + x_2 \leq 1, && \text{(1b)}\\
& 2x_1 + x_2 \leq 4, && \text{(1c)}\\
& x_1 \geq 0. && \text{(1d)}
\end{aligned}
$$

(2p) a) Solve this problem using phase I (so that you begin with a unit matrix as the first basis) and phase II of the simplex method. If the problem has an optimal solution, then present the optimal solution in both the original variables and in the variables used in the standard form. If the problem is unbounded, then use your calculations to find a direction of unboundedness in both the original variables and in the variables used in the standard form.

[*Aid:* The identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

is quite helpful.]

(1p) b) Suppose that the right-hand side of constraint (1b) is changed to $1 + \varepsilon$, where $\varepsilon > 0$ is very small. How much is then the optimal objective value of (1) changed?

[*Hint:* You should use the optimal dual variables from a).]

Question 2

(descent methods in unconstrained optimization)

(1p) a) Consider the unconstrained optimization problem to minimize f over $\mathbb{R}^n$, where $f \in C^1$ on $\mathbb{R}^n$. Suppose, at an iteration k, that the iterate is $\boldsymbol{x}_k$ and that the next iterate $\boldsymbol{x}_{k+1}$ has been produced through an exact line search along the direction of $\boldsymbol{p}_k$.

Establish that $\nabla f(\boldsymbol{x}_{k+1})^{\mathrm{T}} \boldsymbol{p}_k = 0$ holds.

(2p) b) Rosenbrock's famous banana function has the form $f(\boldsymbol{x}) := 100(x_2 - x_1^2)^2 + (1 - x_1)^2$. Verify that the minimizer $\boldsymbol{x}^* = (1,1)^{\mathrm{T}}$ satisfies that $\nabla f(\boldsymbol{x}^*) = \mathbf{0}^2$ and $\nabla^2 f(\boldsymbol{x}^*)$ is positive definite. Further, show that $\nabla^2 f(\boldsymbol{x})$ is singular if and only if $\boldsymbol{x}$ satisfies the condition that $x_2 - x_1^2 = 0.005$, and that $\nabla^2 f(\boldsymbol{x})$ is positive definite for all $\boldsymbol{x} \in \mathbb{R}^2$ such that $f(\boldsymbol{x}) < 0.005$.

(3p) Question 3

(separation and projection)

Let $S \subseteq \mathbb{R}^n$ be a closed convex set, and let $\mathrm{proj}_{\mathrm{S}}(\cdot)$ denote the projection operator onto S.

Let $\boldsymbol{x} \in S$, $\boldsymbol{z} \notin S$. Show that the inequality $\|\mathrm{proj}_{\mathrm{S}}(\boldsymbol{z}) - \boldsymbol{x}\| \leq \|\boldsymbol{z} - \boldsymbol{x}\|$ holds.

[*Hint: if* $\widehat{\boldsymbol{z}} = \mathrm{proj}_{\mathrm{S}}(\boldsymbol{z})$, *then* $\{\, \boldsymbol{y} \in \mathbb{R}^n \,|\, (\boldsymbol{z} - \widehat{\boldsymbol{z}})^{\mathrm{T}}(\widehat{\boldsymbol{z}} - \boldsymbol{y}) = 0 \,\}$ *is a separating hyperplane.*]

Question 4

(true or false claims in optimization)

For each of the following three claims, your task is to decide whether it is true or false. Motivate your answers.

(1p) a) Consider the problem to

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{q}^{\mathrm{T}}\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \end{aligned}$$

where $\boldsymbol{q} \in \mathbb{R}^n$, $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$, that is, a quadratic minimization over a subspace.

Claim: any local minimum is a global minimum.

(1p) b) Suppose a function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a vector $\boldsymbol{x} \in \mathbb{R}^n$.

Claim: for the vector $\boldsymbol{p} \in \mathbb{R}^n$ to be a descent direction with respect to f at $\boldsymbol{x}$ it is necessary that $\nabla f(\boldsymbol{x})^{\mathrm{T}}\boldsymbol{p} < 0$.

(1p) c) In linear programming, if there exists a feasible solution then there exists a finite optimal solution.

Question 5

(Lagrangian duality)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := (x_1 - 2)^2 + (x_2 - 1)^4, \\ \text{subject to} \quad & (x_1 + 3)^2 + (x_2 + 1)^2 \leq 2, \\ & x_1 - x_2 \leq 0. \end{aligned}$$

(2p) a) Use Lagrangian duality to give a lower and upper bound for the optimal value of this problem.

(1p) b) Does strong duality (with respect to the relaxation of *all* constraints) hold for this problem? Motivate why/why not.

(3p) Question 6

(sufficiency of the KKT conditions under convexity)

Consider the problem to find

$$\begin{aligned} f^* := \operatorname*{infimum}_{x} \quad & f(\boldsymbol{x}), \\ \text{subject to} \quad & g_i(\boldsymbol{x}) \leq 0, \qquad i = 1, \ldots, m, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are given differentiable and convex functions. State the KKT conditions for this problem, and assume that a vector $\boldsymbol{x}^*$ satisfies them. Establish that $\boldsymbol{x}^*$ then is a global optimum.

(3p) Question 7

(modelling)

In an experiment, we have measured the velocity and the acceleration of a vehicle at n different time points. Let (t_i, v_i, a_i), $i = 1, \dots, n$, be the measurements, where t_i is the time of measurement i, v_i is the velocity of the vehicle at time t_i, and a_i is the acceleration of the vehicle at time t_i.

We would like to find a function $v(t) = k_1 + k_2 \sin(t) + k_3 \cos(t)$ that approximates the velocity of the vehicle at time t.

Construct a linear optimization model for finding the function $v(t)$ (i.e., the parameters k_1, k_2, k_3) that minimizes the sum of the errors of the velocities and the accelerations at the measures points.

Note: The error of the velocity at point i is $|v(t_i) - v_i|$ and the error of the acceleration at point i is $|v'(t) - a_i|$.