TMA947 / MMG621 — Nonlinear optimisation

Exercise 2 – Convexity

Emil Gustavsson, Michael Patriksson, Adam Wojciechowski

November 5, 2012

E2.1 (easy) Consider the polytope

$$\begin{cases} x_1 + x_2 \leq 2, \\ 2x_1 + x_2 \geq 2, \\ x_1 - x_2 \leq 1. \end{cases}$$

Use both algebraic and graphical methods to answer the following questions.

(a) Is the point $\boldsymbol{x}^1 = (1, 1)$ an extreme point?

(b) Is the point $\boldsymbol{x}^2 = \frac{1}{2}(3, 2)$ an extreme point?

E2.2 (medium)

(a) Is the function $f(x_1, x_2) = x_1^2 + x_2^2 + 3x_1x_2 + 10x_1 - 11x_2 + 5$ convex?

(b) Is the function $f(\boldsymbol{x}) = \sum_{i=1}^n h_i(x_i)^2$ convex, if $h_i : \mathbb{R} \to \mathbb{R}_+$ and h_i is convex?

(c) Does b) still hold if $h_i : \mathbb{R} \to \mathbb{R}$? If not, give a counterexample.

E2.3 (medium) Let $f_1, \dots, f_k : \mathbb{R}^n \to \mathbb{R}$ be convex functions and let f be defined as $f(\boldsymbol{x}) = \max\{f_1(\boldsymbol{x}), \dots, f_k(\boldsymbol{x})\}$. Show that f is convex. State a similar result for concave functions.

E2.4 (medium) Let S be a nonempty set in $\mathbb{R}^n$ and let $\bar{\boldsymbol{x}} \in S$. Consider the set $C = \{\boldsymbol{y} : \boldsymbol{y} = \lambda(\boldsymbol{x} - \bar{\boldsymbol{x}}),\ \lambda \geq 0,\ \boldsymbol{x} \in S\}$.

(a) Show that C is a cone and interpret it geometrically.

(b) Show that C is convex if S is convex.