TMA947 / MMG621 — Nonlinear optimisation

Exercise 3 – Optimality conditions

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E3.1 (easy) Weierstrass theorem claims that an optimal solution of an continuous objective function always exists if either the feasible set is closed and bounded (compact) or if the set is closed and the objective coercive (i.e. $\lim_{\|\boldsymbol{x}\|\to\infty} f(\boldsymbol{x}) = \infty$ in a minimization problem). Consider the following problems. Does an optimal solution exist. If not, why does the problem violate Weierstrass theorem. (Hint: draw the problems)

(a)

$$\begin{array}{ll} \text{minimize} & x^2 \\ \text{subject to} & 0 < x \leq 1 \end{array}$$

(b)

$$\begin{array}{ll} \text{minimize} & e^x \\ \text{subject to} & x \leq 0 \end{array}$$

(c)

$$\begin{array}{ll} \text{minimize} & x^3 - 64x \\ \text{subject to} & |x| < 10 \end{array}$$

(d)

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & |x| \leq 3 \end{array}$$

where

$$f(x) = \begin{cases} -x^2 & \text{if } |x| < 2, \\ -x & \text{if } |x| \geq 2. \end{cases}$$

E3.2 (easy) Consider the following problems. Does an optimal solution exist. If not, why do the problems violate Weierstrass theorem. (Hint: draw the feasible region of the problems)

(a)

$$\begin{aligned} \text{minimize} \quad & -x - y, \\ \text{subject to} \quad & 0 \geq x, \\ & 0 \geq y, \\ & x^2 + y^2 < 1. \end{aligned}$$

(b)

$$\begin{aligned} \text{minimize} \quad & -x - y, \\ \text{subject to} \quad & x + y \geq 1, \\ & 3x \geq y, \\ & y \geq 2x. \end{aligned}$$

E3.3 (medium) Construct a continuous objective f such that no optimal solution exists to the problem $\min_{\boldsymbol{x} \in X} f(\boldsymbol{x})$. Comment on how Weierstrass is violated.

(a)

$$X = \{x \in \mathbb{R} \mid 0 \leq x < 10 \text{ or } 11 \leq x \leq 12\},$$

(b)

$$X = \{\boldsymbol{x} \in \mathbb{R}^2 \mid 2^2 x_1^2 + 5^2 x_2^2 \leq 10^2\} \backslash \{(1,1)\}.$$

E3.4 (easy) State the set of feasible directions for the feasible set X at the point $\boldsymbol{x}$. (Hint: draw the feasible set)

(a)

$$X = \{\boldsymbol{x} \in \mathbb{R}^2 \mid 3x_1^2 + x_2^3 \leq 1\}, \boldsymbol{x} = (0, 0.99)^T.$$

(b)

$$X = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}, \boldsymbol{x} = (1/\sqrt{2}, 1/\sqrt{2})^T.$$

(c)

$$X = \{\boldsymbol{x} \in \mathbb{R}^2 \mid (x_1 - 1)^2 + x_2^2 \geq 1, (x_1 + 1)^2 + x_2^2 \geq 1\}, \boldsymbol{x} = (0, 0)^T.$$

(d)

$$X = \{\boldsymbol{x} \in \mathbb{R}^2 \mid 2x_1 + 3x_2 = 5\}, \boldsymbol{x} = (1, 1)^T.$$

(e)

$$X = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 = x_2^2\}, \boldsymbol{x} = (1, 1)^T.$$

E3.5 (easy) Is the point $(0,1,0)^T$ an optimal solution to the problem

$$\begin{aligned} \text{minimize} \quad & x^2 + (y-2)^2 + z^2, \\ \text{subject to} \quad & x^2 + y^2 + z^2 \leq 1, \\ & x \geq 0, \\ & y \geq 0, \\ & z \geq 0. \end{aligned}$$

Use the variational inequality (proposition 4.23)!

E3.6 (easy) Solve the unconstrained problem

$$\text{minimize } \boldsymbol{x}^T A \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x},$$

by using the optimality conditions for unconstrained optimization if

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.