TMA947 / MMG621 — Nonlinear optimisation

Exercise 6 – Lagrangian relaxation

Emil Gustavsson, Michael Patriksson, Adam Wojciechowski

October 26, 2012

E6.1 (easy) A problem where an objective function f should be minimized has been analyzed by Lagrangian relaxing some constraints and the dual function $q(\boldsymbol{\mu})$ has been created. An iterative algorithm has produced the following results.

(a) A primal feasible $\boldsymbol{x}^1$ has been found and $f(\boldsymbol{x}^1) = 6$. $\boldsymbol{\mu}^1$ is a vector with positive Lagrangian multipliers and $q(\boldsymbol{\mu}^1) = -2$. What can you say about the optimal objective value f^*?

(b) In the next iteration, two more vectors $\boldsymbol{x}^2$ and $\boldsymbol{\mu}^2$ (feasible) have been evaluated and $f(\boldsymbol{x}^2) = 3$ and $q(\boldsymbol{\mu}^2) = -4$. What can you now say about the optimal objective value f^*?

(c) Finally, we manage to maximize the dual function and $q^* = 3$. What are the conclusions regarding f^*?

E6.2 (easy) Consider the problem (P):

$$
\begin{array}{ll}
\text{minimize} & 4x_1 + 3x_2, \\
\text{subject to} & x_1^2 + x_2^2 \leq 25, \\
& (x_1 - 4)^2 + (x_2 - 1)^2 \geq 9,
\end{array}
$$

The point $\boldsymbol{x}_0 = (-4, -3)^T$ is a KKT to (P).

(a) Is (P) convex? Can you say anything about the optimality of $\boldsymbol{x}_0$?

(b) Remove the second constraint and create a reduced problem (R). $\boldsymbol{x}_0$ is a KKT point in (R). What can you say about the optimality of $\boldsymbol{x}_0$ in (R)? In (P)?

E6.3 (easy) Consider the problem

$$
\begin{array}{ll}
\text{minimize} & x_1^3 x_2 - x_3^2 - x_1 x_3, \\
\text{subject to} & -x_1 + x_2 + x_3^2 \leq 4, \\
& x_1^3 - (x_2^2 - 1) + x_3 \geq 2, \\
& x_1 + x_3 \leq 6, \\
& x_1, x_2, x_3 \geq 0.
\end{array}
$$

Lagrange relax the first two constraints and state the dual function q as a minimization problem.

E6.4 (medium) Consider the problem:

$$
\begin{array}{ll}
\text{minimize} & x_1 - 3x_2, \\
\text{subject to} & -x_1 + x_2 \leq 1, \\
& x_1 + x_2 \leq 4, \\
& x_1, x_2 \geq 0.
\end{array}
$$

Lagrangian relax the first constraint and create the Lagrange function $L(\boldsymbol{x}, \mu)$. Find and plot the Lagrangian dual function $q(\mu)$. What can you say about the optimal value of the primal problem? Can you find the optimal solution $\boldsymbol{x}^*$ to the primal problem?

E6.5 (medium) Consider the problem:

$$\begin{aligned} \text{minimize} \quad & -2x_1 + x_2, \\ \text{subject to} \quad & x_1 + x_2 \leq 3, \\ & \boldsymbol{x} \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

Lagrange relax the inequality constraint. Which of the optimality conditions can not be fulfilled? Calculate the optimality gap $\Gamma = f^* - q^*$.