TMA947 / MMG621 — Nonlinear optimisation

Exercise 4 – Linear programming

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November 25, 2013

E4.1 (medium) Consider the feasible set $\{\boldsymbol{x} \in \mathbb{R}^n | A\boldsymbol{x} \leq \boldsymbol{b}\, x \geq \boldsymbol{0}^n\}$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 3 \\ 1 \\ 1 \\ -1 \end{pmatrix}.$$

Draw the feasible region (preferably in MATLAB[1]). Write the program on standard form and find a BFS corresponding to the extreme point $(1, 1, 1)^T$. Is it degenerate? How many different BFS correspond to this point? Compare to the extreme point $(1, 2, 0)^T$.

E4.2 (easy) Write on standard form

$$\begin{aligned} \text{maximize} \quad & 3x_1 - 6x_2, \\ \text{subject to} \quad & 10x_1 - 3x_2 = 5, \\ & -x_1 - 3x_2 \geq 7, \\ & x_2 \geq 5. \end{aligned}$$

E4.3 (easy) Consider the polyhedron

$$\begin{aligned} x_1 + x_2 &\geq 1, \\ x_1 - x_2 &\leq 1, \\ -x_1 + x_2 &\leq 1, \\ x_1 &\leq 2, \\ x_2 &\leq 2. \end{aligned}$$

Find the BFS which corresponds to the extreme point $(2, 2)^T$. Construct new basic solutions by using four out of the five columns included in the BFS corresponding to $(2, 2)^T$ and one column previously not included. Can you obtain any BFS? Which ones? What does theory say about this? (Hint: use MATLAB or Mathematica to calculate $B^{-1}\boldsymbol{b}$. Note also that a variable has to be included into the basis in order to obtain a non-zero value.)

E4.4 (easy) Solve the following LP grahically.

$$\begin{aligned} \text{minimize} \quad & x_1 + 4x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 4 \\ & x_1 + x_2 \geq 2 \\ & x_1 + 2x_2 \geq 3 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Is the optimal solution a BFS, if so, is it unique?

[1] Introduce an indicator function χ for the polyhedron such that $\chi(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in P$ and $\chi(\boldsymbol{x}) = 0$ otherwise. Use the command isosurface. Do not use to many gridpoints!

E4.5 (easy) Solve the following linear program using Phase I and II of the simplex method.

$$\begin{array}{lrcl} \text{minimize} & z = -2x_1 + x_2 & & \\ \text{subject to} & x_1 - 3x_2 & \leq & -3, \\ & 0 \leq x_1, & & \\ & 0 \leq x_2 & \leq & 2. \end{array}$$

E4.6 (medium) Solve the following linear program using Phase I and II of the simplex method.

$$\begin{array}{lrcl} \text{minimize} & z = x_1 + 2x_2 & & \\ \text{subject to} & 2x_1 - 2x_2 & \leq & -2, \\ & 2x_1 + x_2 & \leq & 2, \\ & x_1 & \in & \mathbb{R}, \\ & x_2 & \geq & 0. \end{array}$$

E4.7 (medium) Consider the application of the simplex method to a general LP and suppose that you, unlike in the standard procedure taught in this course, at some iteration a) choose the entering variable to be a non-basic variable with a negative reduced cost but not having the most negative reduced cost, or b) choose the outgoing variable as a basic variable with the $B^{-1}N_{j^*}$ component > 0 but not fulfilling the minimum ratio test. Which of these choices is a critical mistake?

E4.8 (easy) Solve the following linear program using Phase I and II of the simplex method.

$$\begin{array}{lrcl} \text{minimize} & z = 2x_1 - x_2 + x_3, & & \\ \text{subject to} & x_1 + 2x_2 - x_3 & \leq & 7, \\ & -2x_1 + x_2 - 3x_3 & \leq & -3, \\ & x_1, \quad x_2, \quad x_3 & \geq & 0. \end{array}$$

E4.9 (easy) Solve the following linear program using Phase I and II of the simplex method.

$$\begin{array}{lrcl} \text{minimize} & z = -x_1 + x_2, & & \\ \text{subject to} & -x_1 + 2x_2 & \geq & 1/2, \\ & -2x_1 - 2x_2 & \geq & 1, \\ & x_1 & \in & \mathbb{R} \text{ (free)}, \\ & x_2 & \geq & 0. \end{array}$$

E4.10 (medium) Solve the following linear program using phase I and II of the simplex method.

$$\begin{array}{lrcl} \text{minimize} & z = 2x_1 & & \\ \text{subject to} & x_1 \quad - x_3 & = & 3, \\ & x_1 - x_2 \quad - 2x_4 & = & 1, \\ & 2x_1 \quad + x_4 & \leq & 7, \\ & x_1, \quad x_2, \quad x_3, \quad x_4 & \geq & 0. \end{array}$$

E4.11 (easy) Solve the following linear program using phase I and II of the simplex method.

$$\begin{array}{llrrrl}
\text{minimize} & z = & x_1 + & x_2 & +3x_3, & \\
\text{subject to} & & & -\, x_2 & +3x_3 & \leq -1, \\
& & -2x_1 + & x_2 & -\, x_3 & \leq 1, \\
& & x_1, & x_2, & x_3 & \geq 0.
\end{array}$$

Is the optimal solution unique?

E4.12 (medium) Solve the following linear program:

$$\begin{array}{llrrrl}
\text{minimize} & z = & -x_1 - & x_2, & & \\
\text{subject to} & & -x_1 - & 2x_2 & - x_3 = & 2, \\
& & 3x_1 + & x_2 & & \leq -1, \\
& & & x_2, & x_3 \geq & 0, \\
& & x_1 & & & \in \mathbb{R} \text{ (free)}.
\end{array}$$

Is the optimal solution unique?

E4.13 (easy) Formulate the dual to the following problem

$$\begin{array}{ll}
\text{minimize} & 3x_1 + 2x_2 \\
\text{subject to} & x_1 + 2x_2 \leq 3 \\
& x_1 + x_2 \leq 10 \\
& 5x_1 - x_2 \geq 8 \\
& x_1 \geq 0, \quad x_2 \leq 0.
\end{array}$$

E4.14 (medium)

(a) If an LP primal is infeasible, what can you say about its LP dual?

(b) If the LP primal has an optimal solution with reduced costs strictly greater than zero. What can you say about its LP dual?

(c) If the LP dual is unbounded, what can you say about the LP primal?

(d) According to theorem 10.12 an optimal primal dual pair must satisfy primal feasiblity, dual feasibility and complementarity. Which of these conditions is satisfied during the iteration of the simplex algorithm?

E4.15 (easy) Consider the following LP problem

$$\begin{array}{ll}
\text{minimize} & 9x_1 + 3x_2 + 2x_3 + 2x_2 \\
\text{subject to} & \displaystyle\sum_{i=1}^{4} x_i \geq 1, \\
& 3x_1 - x_2 + 2x_4 \geq 1, \\
& x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \leq 0.
\end{array}$$

(a) Use LP duality and a graphical solution to obtain the optimal objective value z^*.

(b) Use complementary slackness to obtain the optimal solution $\boldsymbol{x}^*$.

E4.16 (medium) Let the matrix constraint matrix

$$A = \begin{pmatrix} \cdots \boldsymbol{a}_1^T \cdots \\ \cdots \boldsymbol{a}_2^T \cdots \\ \vdots \\ \cdots \boldsymbol{a}_n^T \cdots \end{pmatrix}.$$

Consider the relaxation of a standard LP problem $\min\{\boldsymbol{c}^T\boldsymbol{x} | A\boldsymbol{x} \geq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$, where we allow a violation of the constraints, but bound the sum of violations by epsilon.

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \quad \boldsymbol{a}_i^T\boldsymbol{x} \geq b_i - v_i,\ i = 1, \ldots, n, \\ & \displaystyle\sum_{i=1}^{n} v_i \leq \varepsilon, \\ & \boldsymbol{x} \geq 0, \boldsymbol{v} \geq 0. \end{array}$$

Formulate the dual and give it an interpretation.