Lecture 10

LP duality

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Consider the **primal LP** written on standard form:

$$
\begin{aligned}
z^* = \text{infimum} \quad & c^T x, \\
\text{subject to} \quad & Ax = b, \\
& x \geq \mathbf{0}.
\end{aligned} \tag{P}
$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$. The corresponding **dual LP** is

$$
\begin{aligned}
q^* = \text{supremum} \quad & b^T y, \\
\text{subject to} \quad & A^T y \leq c, \\
& y \in \mathbb{R}^m.
\end{aligned} \tag{D}
$$

(P) Minimization problem with n variables and m constraints.

(D) Maximization problem with m variables and n constraints.

- At a **basic feasible solution** (BFS), the variables can be ordered s.t.

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}, \quad A = (B, N), \quad c = \begin{pmatrix} c_B \\ c_N \end{pmatrix},$$

  where x_B are **basic variables** and x_N the **non-basic variables**.

- For a specific **basis matrix** B, we have that

$$\begin{aligned} x_B &= B^{-1} b, \\ x_N &= \mathbf{0}^{n-m} \end{aligned}$$

- The simplex algorithm iteratively changes B by one column until it terminates (either optimality or optimal objective value is $-\infty$).

Introducing a dual vector

- Apply simplex algorithm, and assume an optimal basis B is found
- Optimal basis B means that the **reduced costs are nonnegative**:

$$\tilde{c}_N^T = c_N^T - c_B^T B^{-1} N \geq (\mathbf{0}^{n-m})^T \tag{1}$$

We introduce the (optimal dual) vector

$$y^* := (c_B^T B^{-1})^T \tag{2}$$

- By definition in (2), $b^T y^* = (y^*)^T b = c_B^T (B^{-1} b) = c_B^T x_B = c^T x^*$
- In addition, by optimality (i.e., (1)),

$$\left.\begin{array}{l} c_N^T - (y^*)^T N \geq (\mathbf{0}^{n-m})^T \\ c_B^T - (y^*)^T B = \mathbf{0}^m \end{array}\right\} \implies c^T - (y^*)^T A \geq \mathbf{0}^n$$

Thus, y^* satisfies $A^T y^* \leq c \quad$ and $\quad b^T y^* = c^T x^*$

Lower bounds

In general, for two vectors $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ fulfilling

$$\begin{aligned} Ax &= b, \\ x &\geq \mathbf{0}^n \end{aligned} \qquad \begin{aligned} A^T y &\leq c, \\ y &\in \mathbb{R}^m \end{aligned}$$

That is, x is feasible to (P) and y is feasible to (D). Then we have that

$$c^T x \geq y^T Ax = y^T b = b^T y$$

- The dual objective value for any y feasible to (D) is a lower bound of the primal objective value for any x feasible to (P), including z^* the primal optimal objective value.
- We maximize $b^T y$ (s.t. $A^T y \leq c$) to get the best lower bound.

The dual LP

So we formulate the problem of finding the best lower bound as finding

$$
\begin{aligned}
q^* = \text{supremum} \quad & b^T y, \\
\text{subject to} \quad & A^T y \leq c, \\
& y \in \mathbb{R}^m.
\end{aligned}
\tag{D}
$$

- We will show that by some simple assumptions, we get $q^* = z^*$, meaning that the lower bound is tight.
- (D) is the same problem as the **Lagrangian dual problem** in Lecture 7. You can try deriving (D) from (P) yourself!
- How do we construct dual problems for all forms of LP's, in addition to the standard form (P) and its dual (D)?

Dual problem, general construction

- We want to construct the dual problems for all forms of LP's.
- Let A be the constraint matrix. Let a_i^T denote the i-th row of A, and A_j denote the j-th column of A:

(primal problem) (dual problem)

$$\begin{array}{llll}
\underset{x}{\text{minimize}} & c^T x & & \\
\text{subject to} & a_i^T x \geq b_i, & i \in M_1, \\
 & a_i^T x \leq b_i, & i \in M_2, \\
 & a_i^T x = b_i, & i \in M_3, \\
 & x_j \geq 0, & j \in N_1 \\
 & x_j \leq 0, & j \in N_2 \\
 & x_j \in \mathbb{R}^n, & j \in N_3
\end{array}
\qquad
\begin{array}{lll}
\underset{y}{\text{maximize}} & b^T y & \\
\text{subject to} & y_i \geq 0, & i \in M_1, \\
 & y_i \leq 0, & i \in M_2, \\
 & y_i \in \mathbb{R}^m, & i \in M_3, \\
 & A_j^T y \leq c_j, & j \in N_1, \\
 & A_j^T y \geq c_j, & j \in N_2, \\
 & A_j^T y = c_j, & j \in N_3.
\end{array}$$

$$
\begin{aligned}
z^* = \text{infimum} \quad & c^T x, \\
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& y \in \mathbb{R}^m.
\end{aligned} \tag{D}
$$

We will now present some theoretical results about this primal/dual pair.

Weak duality theorem

If x is a feasible solution to (P) and y is a feasible solution to (D), then

$$c^T x \geq b^T y.$$

Proof: Primal feasibility and dual feasibility implies

$$Ax = b \tag{3a}$$

$$x \geq \mathbf{0} \tag{3b}$$

$$c \geq A^T y \tag{3c}$$

Thus,

$$c^T x \overset{(3c),(3b)}{\geq} y^T A x \overset{(3a)}{=} b^T y.$$

Weak duality, II

Corollary

- If the optimal objective value of primal problem (P) is $-\infty$, then dual problem (D) is infeasible.
- If the optimal objective value of dual problem (D) is $+\infty$, then primal problem (P) is infeasible.

Proof: If (D) is feasible, then there exists y s.t. $A^T y \leq c$ and $c^T x \geq b^T y > -\infty$ for all x feasible to (P). Thus, the optimal objective value of primal problem (P) is bounded from below. This shows the first statement. The second statement can be shown similarly.

Weak duality, III

Corollary

If x is a feasible solution to (P), y is a feasible solution to (D), and

$$c^T x = b^T y,$$

then x is optimal in (P) and y is optimal in (D).

Proof: By statement assumption and weak duality theorem,

$$\begin{aligned} c^T x = b^T y \leq c^T \tilde{x}, &\quad \forall \tilde{x} : A\tilde{x} = b,\ \tilde{x} \geq \mathbf{0}, \\ b^T y = c^T x \geq b^T \tilde{y}, &\quad \forall \tilde{y} : A^T \tilde{y} \leq c \end{aligned}$$

Thus, x is optimal in (P) and y is optimal in (D).

Strong duality

Strong duality theorem

If (P) and (D) both have feasible solutions, then there exist optimal solutions to (P) and (D), and their optimal objective function values are the same.

Proof: (P) and (D) feasible implies simplex algorithm terminates with an optimal basis matrix B with the corresponding BFS x^* optimal to (P).

Define $(y^*)^T = c_B^T B^{-1}$, then $b^T y^* = c_B^T B^{-1} b = c^T x^*$.

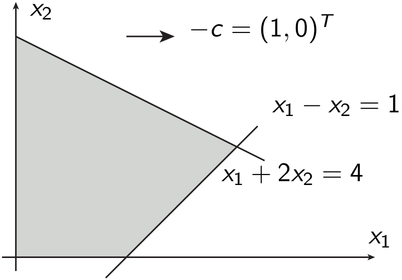
Since B is optimal basis, $c_N^T - c_B^T B^{-1} N \geq \mathbf{0}$. This, together with $c_B^T - c_B^T B^{-1} B = \mathbf{0}$, implies $c^T - (y^*)^T A \geq \mathbf{0}$ and y^* is feasible to (D).

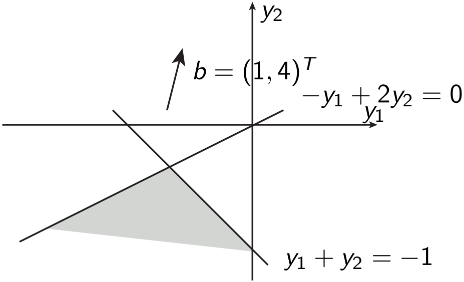
Finally, weak duality theorem implies that y^* is optimal to (D).

Example, strong duality

$$\begin{aligned} \text{minimize} \quad & -x_1, \\ \text{subject to} \quad & x_1 - x_2 \leq 1, \\ & x_1 + 2x_2 \leq 4, \\ & x_1, x_2 \geq 0. \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \text{maximize} \quad & y_1 + 4y_2, \\ \text{subject to} \quad & y_1 + y_2 \leq -1, \\ & -y_1 + 2y_2 \leq 0, \\ & y_1, y_2 \leq 0. \end{aligned} \qquad \text{(D)}$$





Example, strong duality

$$
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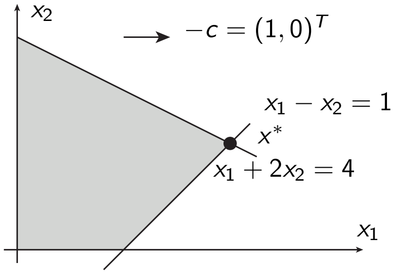
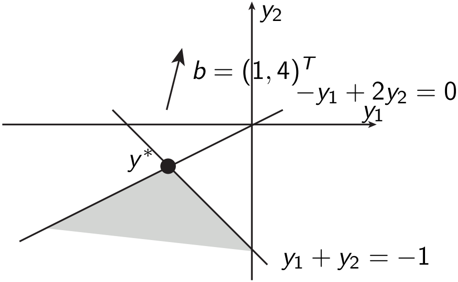
& x_1, x_2 \ge 0.
\end{aligned}
\qquad \text{(P)}
$$

$$
\begin{aligned}
\text{maximize} \quad & y_1 + 4y_2, \\
\text{subject to} \quad & y_1 + y_2 \le -1, \\
& -y_1 + 2y_2 \le 0, \\
& y_1, y_2 \le 0.
\end{aligned}
\qquad \text{(D)}
$$





$c^T x^* = -x_1^* = -2$

$b^T y^* = y_1^* + 4y_2^* = -2$

Obtaining dual solution

The proof of the strong duality theorem was constructive, meaning that we construct an optimal dual solution from an optimal basic feasible solution by

$$(y^*)^T = c_B^T B^{-1}$$

Hence,

- If the primal problem is solved by the simplex algorithm,
- we obtain the optimal dual solution without any additional effort.
- Supervisor's principle. (Give your boss both x^* and y^*)

LP, matrix of possibilities

- For a LP, only three possibilities are allowed:
  1. There is a finite optimal solution.
  2. The optimal objective value is unbounded (e.g., $-\infty$ for minimization problem).
  3. The problem is infeasible.

- A LP and its dual can have the following possibilities:

(D)\(P)	finite optimum	unbounded	infeasible
finite optimum	possible	impossible	impossible
unbounded	impossible	impossible	possible
infeasible	impossible	possible	possible

Complementary slackness

Complementary Slackness Theorem
Let x be feasible in (P) and y feasible in (D). Then

$$\left.\begin{array}{r} x \text{ optimal to (P)} \\ y \text{ optimal to (D)} \end{array}\right\} \iff x_j(c_j - A_{\cdot j}^T y) = 0, \quad j = 1, \dots, n,$$

where $A_{\cdot j}$ is column j of A.

Proof:

$$x_j(c_j - A_{\cdot j}^T y) = 0, \forall j \implies (c^T - y^T Ax) = 0 \implies c^T x = b^T y$$

By weak duality theorem, x is optimal to (P) and y is optimal to (D).
If x is optimal to (P) and y is optimal to (D), strong duality implies

$$c^T x = b^T y \xRightarrow[A^T y \leq c]{x \geq \mathbf{0},\ Ax = b} x_j(c_j - A_{\cdot j}^T y) = 0, \quad j = 1, \dots, m$$

Complementary slackness

Complementary Slackness Theorem

Let x be feasible in (P) and y feasible in (D). Then

$$\left.\begin{array}{r} x \text{ optimal to (P)} \\ y \text{ optimal to (D)} \end{array}\right\} \iff x_j(c_j - A_{\cdot j}^T y) = 0, \quad j = 1, \dots, n,$$

where $A_{\cdot j}$ is column j of A.

For a primal-dual pair of optimal solutions x^*, y^*

- If there is slack in one constraint, then the respective variable in the other problem is zero.
- If a variable is positive, then there is no slack in the respective constraint in the other problem.

Interpreting the dual solution (I)

$$(P): \quad \begin{aligned} \underset{x}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b, \\ & x \geq \mathbf{0}. \end{aligned}$$

Suppose solving (P) leads to optimal basis B by simplex method.
What can we say about the optimal solution to (P'), perturbed problem?

$$(P'): \quad \begin{aligned} v(b') := \underset{x}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b' (= b + \Delta b), \\ & x \geq \mathbf{0}. \end{aligned}$$

- If in (P) $x_B = B^{-1}b > 0$ then for small enough $|\Delta b|$ (i.e., change from b to b'), B remains an optimal basis for (P'):

$$|\Delta b| \text{ small enough} \implies x'_B = B^{-1}b' = x_B + B^{-1}\Delta b \geq 0$$

$$B \text{ opt in } (P) \implies \tilde{c}_N = (c_N^T - c_B^T B^{-1} N)^T \geq 0 \implies B \text{ opt in } (P')$$

Interpreting the dual solution (II)

- As long as B remains optimal basis for (P'), optimal objective value of (P'), which is denoted $v(b')$, is

$$v(b') = c_B^T x_B' = c_B^T B^{-1} b' = (y^*)^T b',$$

y^* is an optimal dual for vector for (P), obtained by B.

$$\left.\begin{array}{l} B \text{ opt basis of non-degenerate BFS in } (P) \\ \Delta b = (b - b') \text{ small enough in magnitude} \end{array}\right\} \implies v(b') = (y^*)^T b'$$

$\implies$ $v(b')$ locally linear near $b' = b$

- If B is degenerate in (P), B need not be optimal basis for (P') for arbitrarily small $|\Delta b|$ and local linearity of $v(b')$ need not hold.

Shadow price theorem

If, for a given vector b, the optimal solution to (P) corresponds to a non-degenerate BFS, then its optimal objective value is differentiable at b, with

$$\frac{\partial v(b)}{\partial b_i} = y_i^*, \quad i = 1, \ldots, m,$$

that is, $\nabla v(b) = y^*$

- So the dual variable y_i^* is called the **shadow price** of constraint i.
- Assuming the optimal basis does not change, then the theorem states that a unit change in the right-hand side b_i would change the optimal value with the amount y_i^*.

Physical meaning of the dual variable

- Assume that a constraint in the primal problem is

$$x_1 + x_2 \geq 4,$$

  which has the physical meaning that we need to fulfill the demand, which is 4 units.

- Assume that we have solved the problem and the optimal objective value (cost) is 10 Skr. We have also obtained an optimal dual variable $y^* = 2$ corresponding to that constraint.

- What happens with the optimal value if we increase our demand to 4.5 units? (Changing the constraint to $x_1 + x_2 \geq 4.5$)

  Assuming that this change of the demand does not change the optimal basis, the change in the optimal objective value would be

$$0.5 \cdot y^* = 1 \ \text{ Skr}$$

- So increasing the demand to 4.5 units would mean that the new optimal objective value (cost) is 11 Skr.

Sensitivity analysis

Cosider the LP

$$\begin{aligned} z^* = \text{infimum} \quad & c^T x, \\ \text{subject to} \quad & Ax = b, \\ & x \geq \mathbf{0}. \end{aligned} \tag{P}$$

We will now study two different perturbations of the LP, namely

- perturbations in the objective function coefficients c_j; and
- perturbations in the right-hand side coefficients b_i.

Assume that we have solved this problem, i.e., we have $x^* = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$

Perturbations in the objective

The objective function is perturbed by the vector $p \in \mathbb{R}^n$, that is

$$\begin{aligned} \bar{z}^* = \text{infimum} \quad & (c + p)^T x, \\ \text{subject to} \quad & Ax = b, \\ & x \geq \mathbf{0}. \end{aligned} \tag{P}$$

- The optimal solution x^* to the original problem is clearly feasible in this problem. But is it still optimal?
- Rearrange $p = (p_B^T, p_N^T)^T$. The optimality condition for the BFS determined by the basis B is that the reduced costs are nonnegative:

$$\tilde{c}_N^T = (c_N + p_N)^T - (c_B + p_B)^T B^{-1} N \geq \mathbf{0}$$

  This is a sufficient condition for x^* still being optimal. Not a necessary condition however.

If only one component of c_N is perturbed, i.e.,

$$p = \begin{pmatrix} p_B \\ p_N \end{pmatrix} = \begin{pmatrix} \mathbf{0}^m \\ \varepsilon e_j \end{pmatrix}, \qquad \varepsilon \in \mathbb{R}$$

Then we have that x^* is optimal in the perturbed problem if

$$\begin{aligned} \tilde{c}_N^T &= (c_N + p_N)^T - (c_B + p_B)^T B^{-1} N \\ &= (c_N + \varepsilon e_j)^T - (c_B)^T B^{-1} N \geq \mathbf{0} \end{aligned}$$

So a change in a non-basic coefficient c_j only affects the j^{th} reduced cost, i.e., we only need to check that

$$(\tilde{c}_N)_j = (c_N)_j + \varepsilon - c_B^T B^{-1} N_j \geq 0$$

Perturbations in objective (basic var)

If only one component of c_B is perturbed, i.e.,

$$p = \begin{pmatrix} p_B \\ p_N \end{pmatrix} = \begin{pmatrix} \varepsilon e_j \\ \mathbf{0}^{n-m} \end{pmatrix}$$

for some $\varepsilon \in \mathbb{R}$. Then we have that x^* is optimal in the perturbed problem if

$$\begin{aligned} \tilde{c}_N^T &= (c_N + p_N)^T - (c_B + p_B)^T B^{-1} N \\ &= c_N^T - (c_B + \varepsilon e_j)^T B^{-1} N \geq \mathbf{0} \end{aligned}$$

So a change in a non-basic coefficient c_j affects all the reduced cost, i.e., we need to check that the whole vector $\tilde{c}_N \geq \mathbf{0}$

Large perturbations in the objective

The objective function is perturbed by the vector $p \in \mathbb{R}^n$, that is

$$\begin{aligned} \bar{z}^* = \text{infimum} \quad & (c+p)^T x, \\ \text{subject to} \quad & Ax = b, \\ & x \geq \mathbf{0}. \end{aligned} \tag{P}$$

- Optimal solution to original problem ($p = 0$) is feasible for (P), since $x_B = B^{-1} b$ independent of p.
- If p is too large, sufficient condition

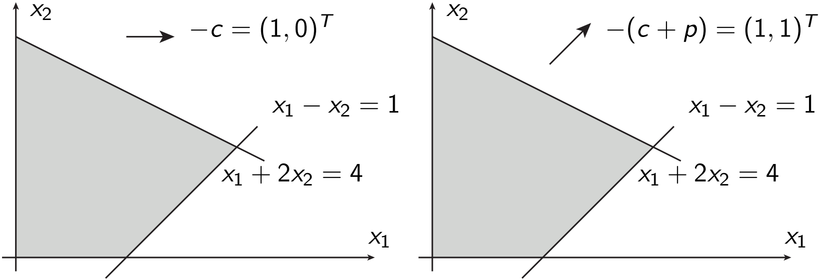
  $$\tilde{c}_N^T = (c_N + p_N)^T - (c_B + p_B)^T B^{-1} N \geq \mathbf{0}$$

  need not hold.
- Start another run of simplex algorithm with starting basis B!

Example, perturbation in objective

Primal problem

Perturbation in the objective by a vector $p = (0, -1)^T$.



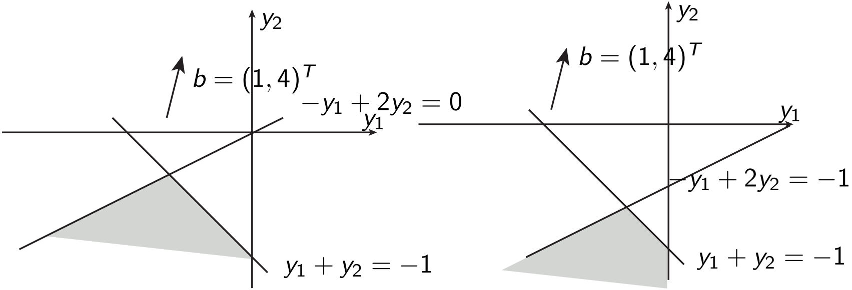
$c = (-1, 0)^T$ $\qquad$ $c + p = (-1, -1)^T$

(We change the negative gradient)

Example, perturbation in objective

Dual problem

Perturbation in the objective by a vector $p = (0, -1)^T$.



$c = (-1, 0)^T$ $\qquad$ $c + p = (-1, -1)^T$

(We change the feasible set)

Perturbations in the right-hand side

The right-hand side is perturbed by a vector $p \in \mathbb{R}^m$.

$$\begin{aligned} \bar{z}^* = \text{infimum} \quad & c^T x, \\ \text{subject to} \quad & Ax = b + p, \\ & x \geq \mathbf{0}. \end{aligned} \tag{P}$$

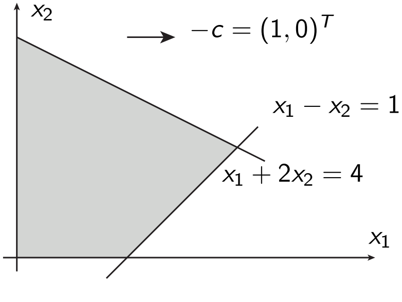
Now the reduced costs do not change, so the optimal solution x^* to the original problem is optimal to the perturbed problem as long as x^* is feasible in the perturbed problem, i.e., if

$$x^* = \begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} B^{-1}(b+p) \\ \mathbf{0}^{n-m} \end{pmatrix} \geq \mathbf{0}^n$$

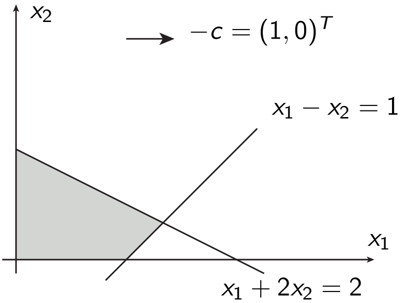
So we need to check that if $B^{-1}(b+p) \geq \mathbf{0}^m$

Primal problem

Perturbation in the right-hand side by a vector $p = (0, -2)^T$.



$c = (-1, 0)^T$

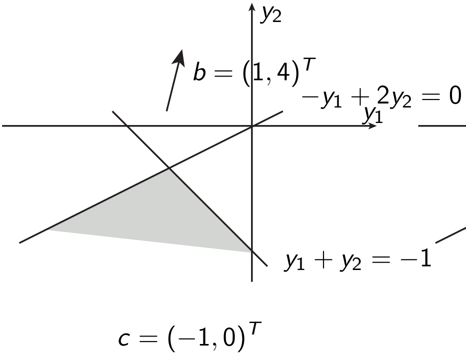


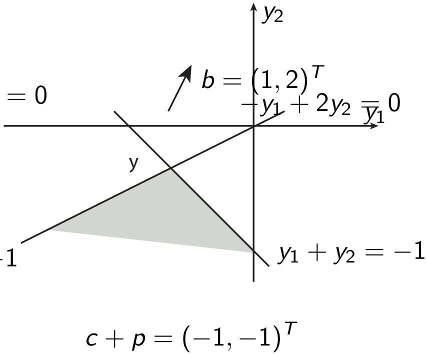
$c + p = (-1, -1)^T$

(We change the feasible set)

Dual problem

Perturbation in the right-hand side by a vector $p = (0, -2)^T$.





(We change the gradient)

Large perturbations in the RHS

The right-hand side is perturbed by a vector $p \in \mathbb{R}^m$.

$$\begin{aligned} \bar{z}^* = \text{infimum} \quad & c^T x, \\ \text{subject to} \quad & Ax = b + p, \\ & x \geq \mathbf{0}. \end{aligned} \tag{P}$$

- Optimal solution to original problem (when $p = 0$) is optimal to (P) if and only if $x_B(p) = B^{-1}(b + p) \geq \mathbf{0}$, since reduced costs $\tilde{c}_N = (c_N^T - c_B^T B^{-1} N)^T$ are independent of p.
- If p is too large, the $x_B(p) \geq \mathbf{0}$ condition need not hold.
- Still, we can run the **dual simplex algorithm** with a starting dual feasible basis (see text).