Chalmers/Gothenburg University
Mathematical Sciences

**EXAM**

# TMA947/MAN280
# OPTIMIZATION, BASIC COURSE

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| **Date:** | 11–12–12 |
| **Time:** | House V, morning |
| **Aids:** | Text memory-less calculator, English–Swedish dictionary |
| **Number of questions:** | 7; passed on one question requires 2 points of 3.<br>Questions are *not* numbered by difficulty.<br>To pass requires 10 points and three passed questions. |
| **Examiner:** | Michael Patriksson |
| **Teacher on duty:** | Adam Wojciechowski (0703-088304) |
| **Result announced:** | 12–01–09<br>Short answers are also given at the end of the exam on the notice board for optimization in the MV building. |

## Exam instructions

### When you answer the questions

*Use generally valid theory and methods.*
*State your methodology carefully.*

*Only write on one page of each sheet. Do not use a red pen.*
*Do not answer more than one question per page.*

### At the end of the exam

*Sort your solutions by the order of the questions.*
*Mark on the cover the questions you have answered.*
*Count the number of sheets you hand in and fill in the number on the cover.*

## Question 1

(the simplex method)

Consider the following linear program:

$$\begin{aligned} \text{maximize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \geq 1, \\ & x_1 - x_2 \geq -2, \\ & x_1 \geq 0, \\ & x_2 \geq 0. \end{aligned}$$

**(2p)** a) Solve this problem using phase I (so that you begin with a unit matrix as the first basis) and phase II of the simplex method.

*Aid:* utilize the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

**(1p)** b) If an optimal solution exists, use your calculations to decide if it unique. If the problem is unbounded, use your calculations to specify the direction of unboundedness.

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**(3p)** 

## Question 2

(modeling)

An airplane has a route that takes it from city 1 to city $n$ by going from city $i$ to $i+1$, where $i = 1, \ldots, n-1$. Let

- $w_i$ be the weight, excluding fuel, of the plane on flight from city $i$ to $i+1$, $i = 1, \ldots, n-1$,
- $c_i$ be the cost of fuel per unit weight at city $i$, $i = 1, \ldots, n$,
- $K_i$ be the maximum amount of fuel that can be purchased in city $i$, $i = 1, \ldots, n$,

- $M$ the maximum weight of fuel that can be loaded into the plane.

Let $z_i$ be the variables denoting the total combined weight of the plane, including fuel, at takeoff from city $i$, $i = 1, \dots, n-1$. Assume that the amount of fuel (in weight units) needed to fly from city $i$ to $i+1$, $i = 1, \dots, n-1$, is $\alpha_i z_i$, where $\alpha_i$ are given positive constants. Formulate a linar program that determines how much fuel one should buy at each city, such that the total fual cost for completing the trip is minimized.

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## Question 3

(interior penalty methods)

Consider the problem to

$$\begin{aligned} &\text{minimize } f(\boldsymbol{x}) := (x_1 - 2)^4 + (x_1 - 2x_2)^2, \\ &\text{subject to } g(\boldsymbol{x}) := x_1^2 - x_2 \leq 0. \end{aligned}$$

We attack this problem with an interior penalty (barrier) method, using the barrier function $\phi(s) = -s^{-1}$. The penalty problem is to

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \; f(\boldsymbol{x}) + \nu \hat{\chi}_S(\boldsymbol{x}), \tag{1}$$

where $\hat{\chi}_S(\boldsymbol{x}) = \phi(g(\boldsymbol{x}))$, for a sequence of positive, decreasing values of the penalty parameter $\nu$.

We repeat a general convergence result for the interior penalty method below.

THEOREM 1 (convergence of an interior point algorithm) *Let the objective function $f : \mathbb{R}^n \to \mathbb{R}$ and the functions $g_i$, $i = 1, \dots, m$, defining the inequality constraints be in $C^1(\mathbb{R}^n)$. Further assume that the barrier function $\phi : \mathbb{R}_- \to \mathbb{R}_+$ is in $C^1$ and that $\phi'(s) \geq 0$ for all $s < 0$.*

*Consider a sequence $\{\boldsymbol{x}_k\}$ of points that are stationary for the sequence of problems (1) with $\nu = \nu_k$, for some positive sequence of penalty parameters $\{\nu_k\}$ converging to 0. Assume that $\lim_{k \to +\infty} \boldsymbol{x}_k = \hat{\boldsymbol{x}}$, and that LICQ holds at $\hat{\boldsymbol{x}}$. Then, $\hat{\boldsymbol{x}}$ is a KKT point of the problem at hand.*

*In other words,*

$$\left.\begin{array}{r} \boldsymbol{x}_k \text{ stationary in } (1) \\ \boldsymbol{x}_k \to \hat{\boldsymbol{x}} \text{ as } k \to +\infty \\ \text{LICQ holds at } \hat{\boldsymbol{x}} \end{array}\right\} \implies \hat{\boldsymbol{x}} \text{ stationary in our problem.}$$

**(1p)** a) Does the above theorem apply to the problem at hand and the selection of the penalty function?

**(2p)** b) Implementing the above-mentioned procedure, the first value of the penalty parameter was set to $\nu_0 = 10$, which is then divided by ten in each iteration, and the initial problem (1) was solved from the strictly feasible point $(0, 1)^{\mathrm{T}}$. The algorithm terminated after six iterations with the following results: $\boldsymbol{x}_6 \approx (0.94389, 0.89635)^{\mathrm{T}}$, and the multiplier estimate [given by $\nu_6 \phi'(g(\boldsymbol{x}_6))$] $\hat{\mu}_6 \approx 3.385$. Confirm that the vector $\boldsymbol{x}_6$ is close to being a KKT point. Are the KKT point(s) globally optimal? Why/Why not?

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## Question 4

(Lagrangian duality)

Consider the quadratic problem

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \end{aligned} \tag{1}$$

where $\boldsymbol{Q}$ is a symmetric matrix.

**(1p)** a) Assume that $\boldsymbol{Q}$ is positive definite. Construct the Lagrangian dual problem by relaxing all the constraints and show that the dual problem itself is a quadratic problem.

*Hint:* An explicit solution to the problem $\min_{x \in X} L(\boldsymbol{x}, \boldsymbol{\mu})$ can be found for each $\boldsymbol{\mu}$.

**(1p)** b) Is the dual function always strictly concave if $\boldsymbol{Q}$ is positive definite? If so, provide a proof. If not, provide a counter example.

**(1p)** c) Consider the following properties:

i) the primal is a convex problem;
ii) the dual is a convex problem;
iii) the dual objective function $q$ is differentiable;
iv) the duality gap is zero (i.e. $q^* = f^*$).

Which of these hold when $\boldsymbol{Q}$ is positive definite? Which properties do the primal and dual problems have when $\boldsymbol{Q}$ has a negative eigenvalue? Motivate your answers!

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## (3p) Question 5

(optimality conditions)

Farkas' Lemma can be stated as follows:

*Let $\boldsymbol{A}$ be an $m \times n$ matrix and $\boldsymbol{b}$ an $m \times 1$ vector. Then exactly one of the systems*

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{b}, \\ \boldsymbol{x} &\geq \boldsymbol{0}^n, \end{aligned} \tag{I}$$

*and*

$$\begin{aligned} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{y} &\leq \boldsymbol{0}^n, \\ \boldsymbol{b}^{\mathrm{T}}\boldsymbol{y} &> 0, \end{aligned} \tag{II}$$

*has a feasible solution, and the other system is inconsistent.*

Prove Farkas' Lemma.

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## (3p) Question 6

(LP duality)

Consider the problem to

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}^n, \end{aligned} \tag{P}$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{c} \in \mathbb{R}^n$ are given matrices.

Assume that the program (P) has multiple optimal solutions. You are therefore interested in finding an optimal solution to (P) that has the minimum value with respect to another linear objective function, $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{x}$. Formulate a linear program which will yield such an optimal solution, without first solving the problem (P).

*Hint:* There is a means to describe the set of primal–dual optimal solutions to (P) as a system of linear inequalities.

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## (3p) Question 7

(sequential linear programming)

Consider the following nonlinear programming problem: find $\boldsymbol{x}^* \in \mathbb{R}^n$ that solves the problem to

$$\begin{aligned} \text{minimize} \ & f(\boldsymbol{x}), && && \text{(1a)}\\ \text{subject to} \ & g_i(\boldsymbol{x}) \leq 0, && i = 1,\dots,m, && \text{(1b)}\\ & h_j(\boldsymbol{x}) = 0, && j = 1,\dots,\ell, && \text{(1c)} \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g_i$, and $h_j : \mathbb{R}^n \to \mathbb{R}$ are given functions in $C^1$ on $\mathbb{R}^n$.

We are interested in establishing that the classic Sequential Linear Programming (SLP) subproblem tells us whether an iterate $\boldsymbol{x}_k \in \mathbb{R}^n$ satisfies the KKT conditions or not, thereby establishing a natural termination criterion for the SLP algorithm.

Given the feasible iterate $\boldsymbol{x}_k$, the SLP subproblem is to

$$\begin{aligned} \underset{p}{\text{minimize}} \ & \nabla f(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p}, && && \text{(2a)}\\ \text{subject to} \ & g_i(\boldsymbol{x}_k) + \nabla g_i(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p} \leq 0, && i = 1,\dots,m, && \text{(2b)}\\ & h_j(\boldsymbol{x}_k) + \nabla h_j(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{p} = 0, && j = 1,\dots,\ell, && \text{(2c)}\\ & -1 \leq p_s \leq 1, && s = 1,\dots,n. && \text{(2d)} \end{aligned}$$

This subproblem is natural: it is based on a linearization of both the objective function and the constraint functions, whereby it resembles the Frank–Wolfe method. The main difference, of course, is that the problem (1) has general and perhaps nonlinear constraints which in the subproblem (2) therefore are replaced by first-order Taylor approximations.

Establish the following statement: the vector $\boldsymbol{x}_k$ is a KKT point in the problem (1) if and only if $\boldsymbol{p} = \mathbf{0}^n$ is a globally optimal solution to the SLP subproblem (2). In other words, the SLP algorithm terminates if and only if $\boldsymbol{x}_k$ is a KKT point in the original problem (1).

*Hint:* Compare the KKT conditions of (1) and (2).