Chalmers/Gothenburg University
Mathematical Sciences

EXAM

TMA947/MAN280
OPTIMIZATION, BASIC COURSE

Date:	12–04–10
Time:	House V, morning, 8^{30}–13^{30}
Aids:	Text memory-less calculator, English–Swedish dictionary
Number of questions:	7; passed on one question requires 2 points of 3. Questions are *not* numbered by difficulty. To pass requires 10 points and three passed questions.
Examiner:	Michael Patriksson
Teacher on duty:	Hossein Raufi (0703-088304)
Result announced:	12–04–25 Short answers are also given at the end of the exam on the notice board for optimization in the MV building.

Exam instructions

When you answer the questions

Use generally valid theory and methods.
State your methodology carefully.

Only write on one page of each sheet. Do not use a red pen.
Do not answer more than one question per page.

At the end of the exam

Sort your solutions by the order of the questions.
Mark on the cover the questions you have answered.
Count the number of sheets you hand in and fill in the number on the cover.

Question 1

(the simplex method)

Consider the following linear program:

$$\begin{aligned} \text{minimize} \quad & 2x_1 - 2x_2, \\ \text{subject to} \quad & -x_1 + 2x_2 \geq 2, \\ & -x_1 + x_2 \leq 3, \\ & x_2 \geq 0. \end{aligned}$$

(2p) a) Solve this problem using phase I (so that you begin with a unit matrix as the first basis) and phase II of the simplex method.

Aid: Utilize the identity

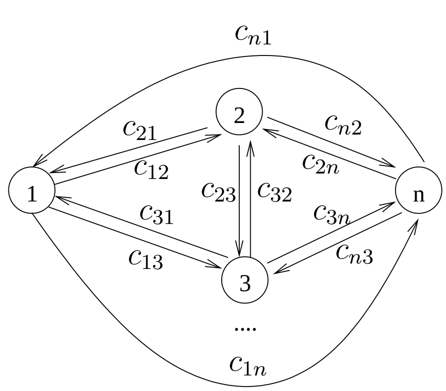
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(1p) b) If an optimal solution exists, use your calculations to decide if it is unique. If the problem is unbounded, use your calculations to specify a direction of unboundedness of the objective value.

(3p) Question 2

(LP duality)

Consider the following complete graph on n nodes:



Let the set $A = \{ (i,j) \in \{1,\dots,n\} \times \{1,\dots,n\} \mid i \neq j \}$ be the set of all arcs in the above graph. Assume given constants $c_{ij} \geq 0$ for each arc $(i,j) \in A$. Introduce variables x_{ij} for all arcs $(i,j) \in A$ and an additional variable s, and consider the following LP problem:

$$\begin{aligned} \text{maximize} \quad & s, \\ \text{subject to} \quad & \sum_{\substack{i=1 \\ i\neq j}}^{n} x_{ij} - \sum_{\substack{i=1 \\ i\neq j}}^{n} x_{ji} = \begin{cases} -s, & j = 1, \\ 0, & j \in \{2,\dots,n-1\}, \\ s, & j = n, \end{cases} \\ & 0 \leq x_{ij} \leq c_{ij}, \qquad (i,j) \in A. \end{aligned} \tag{1}$$

i) Interpret the LP (1) in terms of the graph, what does it mean?

ii) Find the LP dual to (1).

iii) Interpret the LP dual problem in terms of the graph.

Hint: The concept of a *cut* is important when interpreting the dual. A cut between node $i \in \{1,\dots,n\}$ and node $j \in \{1,\dots,n\}$ corresponds to a set of arcs, such that by removing these, no path between i and j exists in the graph (the graph is cut into two parts).

Question 3

(modeling)

Consider K convex polyhedra $P^1, P^2, \dots, P^K$ in $\mathbb{R}^n$ described by

$$P^k = \{ \boldsymbol{x} \in \mathbb{R}^n \mid A^k \boldsymbol{x} \leq \boldsymbol{b}^k \}, \quad \text{where } A^k \in \mathbb{R}^{m_k \times n},\ \boldsymbol{b}^k \in \mathbb{R}^{m_k} \quad \text{for } k = 1,\dots,K.$$

The *convex hull* of a set P is defined as the smallest convex set containing P and is denoted by $\operatorname{conv}(P)$.

(1p) a) Formulate a linear optimization model for minimizing the objective function $\boldsymbol{c}^T\boldsymbol{x}$ over the set

$$\operatorname{conv}\Big(\bigcap_{k=1}^{K} P^k \Big).$$

(2p) b) Formulate an optimization model for minimizing the objective function $\boldsymbol{c}^T\boldsymbol{x}$

over the set

$$\mathrm{conv}\Big(\bigcup_{k=1}^{K} P^k\Big).$$

The model should only contain continuous variables and continuous constraints. It does however not need to be linear.

Question 4

(exterior penalty method)

Consider the problem to

$$\begin{aligned} &\text{minimize} \quad && f(\boldsymbol{x}) := x_1^2 + x_2^2, \\ &\text{subject to} \quad && h(\boldsymbol{x}) := x_1 + x_2 - 1 = 0. \end{aligned} \tag{1}$$

We consider solving (1) by using the external penalty method with the quadratic penalty function $\psi(s) := s^2$, $s \in \mathbb{R}$. The penalty problem is to

$$\underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{minimize}} \quad f(\boldsymbol{x}) + \nu \hat{\chi}_S(\boldsymbol{x}),$$

where $\hat{\chi}_S(\boldsymbol{x}) = \psi(h(\boldsymbol{x}))$, for positive, increasing values of the penalty parameter ν.

(1p) a) By applying the KKT conditions to problem (1), determine its optimal solution and the corresponding optimal Lagrange multiplier.

(1p) b) Apply the exterior penalty method for the problem (1), and show that the sequence of (explicitly stated) solutions to the penalty problem converges to the unique primal solution when $\nu \to \infty$.

(1p) c) Provide the corresponding sequence of estimates of the Lagrange multiplier, and show that it converges to the optimal Lagrange multiplier.

Question 5

(linear programming: existence of optimal solutions)

Let $P := \{ \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{Ax} = \boldsymbol{b};\ \boldsymbol{x} \geq \boldsymbol{0}^n \}$ and $V := \{\boldsymbol{v}^1, \dots, \boldsymbol{v}^k\}$ be the set of extreme points of P. Further, let $C := \{ \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{Ax} = \boldsymbol{0}^m;\ \boldsymbol{x} \geq \boldsymbol{0}^n \}$ and $D := \{\boldsymbol{d}^1, \dots, \boldsymbol{d}^r\}$ be the set of extreme directions of C.

Consider the linear program

$$\begin{aligned} \text{minimize} \quad & z = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}, \\ \text{subject to} \quad & \boldsymbol{x} \in P. \end{aligned} \tag{1}$$

The task is to establish the following two basic facts about solutions to the basic linear program (1).

(2p) a) This problem has a finite optimal solution if and only if P is nonempty and z is lower bounded on P, that is, if P is nonempty and $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{d}^j \geq 0$ for all $\boldsymbol{d}^j \in D$.

(1p) b) If the problem has a finite optimal solution, then there exists an optimal solution among the extreme points.

Question 6

(basic facts in optimization)

The below three statements should be assessed. Are they true or false? Provide an answer together with a short but complete motivation.

(1p) a) For the phase I problem of the simplex method (when a BFS is *not* known a priori), the optimal value is always zero.

(1p) b) Suppose $f \in C^2$. If, at some iteration point $\boldsymbol{x} \in \mathbb{R}^n$ there exists a solution $\boldsymbol{p}$ to the search direction-finding problem of Newton's method then it defines a descent direction for f at $\boldsymbol{x}$.

(1p) c) Consider the convex program

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}), \\ \text{subject to} \quad & g_i(\boldsymbol{x}) \leq 0, \quad i = 1, \dots, m, \end{aligned}$$

where the functions f and g_i, $i = 1, \dots, m$, are convex. Suppose that $\boldsymbol{x}^*$ is a globally optimal solution to this problem, and that $g_k(\boldsymbol{x}^*) < 0$ for some index $k \in \{1, \dots, m\}$. Then, if we remove constraint k from the problem its set of optimal solutions is unchanged.

Question 7

(Lagrangian duality)

Consider the strictly convex quadratic optimization problem to

$$\begin{aligned} \text{minimize} \quad & f(\boldsymbol{x}) := \frac{1}{2}(x_1 - 5)^2 + \frac{1}{2}(x_2 - 3)^2, && \text{(1a)} \\ \text{subject to} \quad & x_1 + x_2 \le 5, && \text{(1b)} \\ & 0 \le x_j \le 3, \quad j = 1, 2. && \text{(1c)} \end{aligned}$$

(1p) a) Establish that the optimal solution to this problem is $\boldsymbol{x}^* = (3, 2)^{\mathrm{T}}$, by utilizing the Karush–Kuhn–Tucker conditions.

(2p) b) Do the following:

[1] Explicitly state its Lagrangian dual function q and its Lagrangian dual problem, associated with the Lagrangian relaxation of the constraint (1b).

[2] Solve this Lagrangian dual problem and provide the optimal Lagrange multiplier μ^*. Confirm that this Lagrange multiplier equals the KKT multiplier corresponding to the inequality (1b), from a).

[3] Prove that strong duality holds, that is, prove that $q(\mu^*) = f(\boldsymbol{x}^*)$ holds.
